Equilibrium of Financial Derivative Markets and Compensating Variations under Portfolio Insurance Constraints*

Philippe Bertrand
GREQAM, University of Aix-Marseille 2
and
Euromed Management
Tel : 33 (0)4 91 14 07 43
e-mail: philippe.bertrand@univmed.fr

Jean-luc Prigent
THEMA, University of Cergy-Pontoise,
33, Bd du Port, 95011, Cergy-Pontoise, France
Tel : 331 34 25 61 72; Fax: 331 34 25 62 33
e-mail: jean-luc.prigent@u-cergy.fr

This version: June 2011

Abstract

This paper examines the equilibrium of portfolio under insurance constraints on the terminal wealth. We consider a single period economy in which agents search to maximize the expected utilities of their terminal wealths. Three main classes of financial assets are considered: a riskless asset (usually the bond), a risky asset (the stock) and European options of all strikes (corresponding to financial derivatives). Both partial and general optimal financial equilibria are determined and analyzed for quite general utility functions and insurance constraints. We introduce also the notion of compensating variation to quantify the monetary loss of not having the true optimal portfolio profile, for the clients and also for the bankers.

Key words: Optimal positioning; financial derivatives; portfolio insurance; financial equilibrium; compensating variation.

AMS 2000 classification: 91B16, 91B52, 91B70, 91G10.

JEL classification: C62, G11, L10.

*We gratefully acknowledge l'Observatoire de l' Epargne Européenne (OEE) for its financial support.

1 Introduction

The structured financial products have been introduced to enhance portfolio returns. The demand for such products has quickly increased. They can allow investors to benefit from the risky asset rises, while being exposed only partially to market drops. The combination of basic assets gives birth to new assets with very specific characteristics whose evaluation appears very complex. During the periods of financial markets decline and strong volatilities, the demand in favour of the structured products in particular those with a protection clause on capital growths sharply.[1] Portfolio insurance payoff provides for a benefit payable at maturity. It is designed to give the investor the ability to limit downside risk while allowing some participation in upside markets. Such methods allow investors to recover, at maturity, a given percentage of their initial capital, in particular in falling markets. This payoff is a function of the value at maturity of some specified portfolio of common assets, usually called the benchmark. As well-known by practitioners, specific insurance constraints on the horizon wealth must be generally satisfied. For example, a minimum level of wealth and some participation in the potential gains of the benchmark can be guaranteed. However, institutional investors for instance may require more complicated insurance contracts.

The two main standard portfolio insurance strategies are the *Constant Proportion Portfolio Insurance* (CPPI) and the *Option Based Portfolio Insurance* (OBPI). The CPPI has been introduced by Perold (1986) for fixed-income instruments and Black and Jones (1987) for equity instruments (see also Perold and Sharpe, 1988). This portfolio strategy is based on a dynamic portfolio allocation during the whole management period. The investor begins by choosing a floor equal to the lowest acceptable value of his portfolio value. Then, at any time, the amount (called the exposure) invested on the risky asset, is proportional to the excess of the portfolio value over the floor, usually called the cushion. The remaining funds are invested in cash, usually T-bills. The proportional factor is defined as the multiple. Both floor and multiple depend on the investor's risk tolerance and are exogenous to the model. This portfolio strategy implies that, if the cushion value converges to zero, then exposure approaches zero too. In continuous time, this prevents portfolio value from falling below the floor, except if there is a very sharp drop in the market before the investor can modify his portfolio weights.

The optimality of such dynamic portfolio strategies can be based on the literature on general portfolio optimization. In this framework, generally we

[1] Indeed, the risk aversion plays a crucial role in the investors behavior. Taking account of the investors psychology, of their cognitive biases and emotional reactions, behavioral finance provides a specific framework for the study of these products. Thus several studies have been published on this research topic. For instance, Hens and Riger (2008) prove that the investor will include more complex structured products than standard equities in his portfolio. Driessen and Maenhout (2007) deal with optimal positioning problems, assuming either expected utility or the CPT of Tversky and Kahneman (1992).

consider an investor who maximizes the expected utility of his terminal wealth, by trading in continuous time (see for example Cox and Huang, 1989; Cvitanic and Karatzas, 1996). The continuous-time setup is also usually introduced to study portfolio insurance (see for example, Grossman and Vila, 1989, Basak, 1995, and Grossman and Zhou, 1996). The key assumption is that markets are complete: all portfolio profiles at maturity are perfectly hedgeable.

The OBPI, introduced by Leland and Rubinstein (1976), consists of a portfolio invested in a risky asset S (usually a financial index such as the $S\&P$) covered by a listed put written on it. Whatever the value of S at the terminal date T, the portfolio value will be always greater than the strike K of the put. At first glance, the goal of the OBPI method is to guarantee a fixed amount only at the terminal date. In fact, if the financial market is frictionless, the OBPI method allows one to get a portfolio insurance at any time. The OBPI is a particular case of the optimal positioning problem which has been addressed in the partial equilibrium context by Brennan and Solanki (1981) and by Leland (1980). More generally, the literature about welfare gains by introducing options into a economy has been initiated by Ross (1976) and extended by Hakansson (1978), Breeden and Litzenberger (1978), Friesen (1979), Arditti and John (1980) and Kreps (1982). The optimal design of optimal contracts has been also further studied by Johnston and McConnell (1989) and Duffie and Jackson (1989).

The value of the portfolio is a function of the benchmark, in a one period set up. An optimal payoff, maximizing the expected utility, is derived. It is shown that it depends crucially on the risk aversion of the investor. Following this approach, Carr and Madan (2001) consider markets in which exist out-of-the-money European puts and calls of all strikes. As they mentioned, this assumption allows to examine the optimal positioning in a complete market and is the counterpart of the assumption of continuous trading. This approximation is justified when there is a large number of option strikes (e.g. for the S&P500, for example). Due to practical constraints, liquidity, transaction costs..., portfolios are in fact discretely rebalanced. Such type of insurance strategy corresponds to optimal portfolio strategies, under specific assumptions, as proved by Bertrand *et al.* (2001a).[2]It can be shown that the optimal payoff (maximizing the expected utility) depends crucially on the risk aversion and prudence of the investor (see e.g. Eeckhoudt and Gollier, 2005; Bertrand and Prigent, 2010).

In both previous cases, only one type of economic agent is considered: the buyer of portfolio insurance. But, who should buy and who should sell insured portfolios? What is the impact of portfolio insurance on financial markets and economies? Such important questions have been partially examined, through equilibrium approach. They constitute the third main part of the research on

[2] Note that, in continuous-time, El Karoui, Jeanblanc and Lacoste (2005) prove that, under a fixed guarantee at maturity, the Option Based Portfolio Strategy (OBPI) is optimal for quite general utility functions (see also Jensen and Sorensen (2001) for a particular case).

portfolio insurance. The study of the general equilibrium model of portfolio insurance has been examined by Basak (1995, 2002); Grossman and Zhou (1996); Carr and Madan (2001). The usual debate about the effects of PI on financial market dynamics is that they may affect market volatility and risk premium. If the efficient market assumption is that equity's volatility is only due to information flow, many practitioners and researchers argue that dynamic trading strategies can increase stock market volatility, in particular PI method (see the Brady report, 1988, about the market crash of October 1987). Brennan and Schwartz (1989), Grossman and Zhou (1996) conclude that market volatility is increased by PI, while, according to Basak (1995, 2002), market volatility is decreased by PI. One explanation of these opposite findings is the different assumptions about agent consumption: for example, Grossman and Zhou (1996) assume that consumption takes place only at PI horizon; Basak (1995) supposes that agents consume continuously. Furthermore, Basak (2002) proves that general equilibrium conditions depend on assumptions about pure exchange or production-type economies. For pure-exchange case, the market price always increases whereas for the production case, the impact is state-dependent. Thus, conclusions about PI and market dynamics (volatility and risk premium) are rather mitigated.

More generally, concerning specifically the evaluation of the risk premium, another stream of literature has recently emerged: the empirical evaluation of the fair pricing of structured products. The analysis of the fair pricing of structured products aims at determining whether financial institutions benefit from an additional premium with respect to a "fair value" when issuing structured products and what is the size of this "excess" gain (between 1% and 5% or beyond 10% as suggested by those who are critic about structured products valuation?). This problem is rather involved, since we have to take account of different specifications: types of the products (complexity and large diversity: see Das, 2000); impact of financial market parameters such as the implied volatility; issuers (retail or private banks for instance)... For example, Stoimenov and Wilkens (2005) examine the pricing of equity-linked structured products in the German market. Using daily closing prices of a large variety of structured products, they compare their actual values to theoretical ones derived from the prices of options traded on the Eurex (European Exchange). They conclude that, for most of the products, large implicit premiums are charged by the issuers.

In this paper, first we determine the optimal financial equilibrium in the optimal positioning framework, under insurance constraints. Then, we investigate if observed risk premium are too high, according to assumptions on financial parameters. It can be argued that holders of insured portfolios are less exposed to bearish markets than the issuers. First, in the partial equilibrium framework, we analyze how investor attitudes towards risk determine what kind of insurance is optimal for investors. Under a variety of modelling strategies, we determine financial general equilibrium and optimal consumption-portfolio-wealth. Then we analyze the competitive price of portfolio insurance, especially when additional guarantee constraints on both clients and issuers (the bankers) are introduced through for instance risk management control. In this framework, we examine whether risk premium of insured products increase under some given solvency regulatory rules. To measure the premium, we use a quantitative measure introduced by de Palma *et al.* (2009) and de Palma and Prigent (2008, 2009) to determine the monetary loss of the investor when the true optimal portfolio is not provided to the investor or to measure the monetary loss of the bank when it must provide a portfolio that meets exactly the investor's preferences. It is based on the standard economic concept of *compensating variation (CV).* If an investor with risk aversion γ and initial investment V_0 can buy his optimal portfolio, his expected utility is $\mathbb{E}[U_\gamma(V_T^*); V_0]$. If this investor selects an optimal portfolio among only those available, then he will get the expected utility $\mathbb{E}[U_\gamma(V_T^{*(\lambda)}); V_0]$. He will get the same expected utility provided that he invests an initial amount $\tilde{V}_0 \geq V_0$.

The paper is organized as follows. In Section 2, we introduce the modelling of the financial market and provide the optimal portfolio profiles with and without insurance constraints. We examine structured portfolios with payoffs defined as functions of the risky asset (a financial stock index for example). An extension of Carr and Madan (2001) is given by introducing insurance constraints on the horizon wealth. Besides, markets can be incomplete. The insured optimal portfolio is characterized for arbitrary utility functions, return distributions and for any choice of a particular risk neutral probability if the market is incomplete.[3] Basic examples are examined. In particular, the optimal portfolio is calculated for CRRA utility functions. In Section 3, we determine the optimal insured portfolio profiles in a general equilibrium framework. We provide in particular the equilibrium risk-neutral probability. Section 4 deals with compensating variations that allow to measure the insurance premium. We illustrate this approach for two kinds of agents: the bankers and their clients. We provide the numerical illustration of the theoretical solutions for banker and investor having both CRRA utilities.[4] We deal with three main cases: for the first one, the investor is assumed to have no direct access to the financial option market. The resulting suboptimality of his standard buy-and-hold portfolio may lead him to bear (theoretically) an additional cost to can include derivatives in his portfolio in order to better fit his true optimal portfolio profile. The second case examines the banker's compensating variation due to more risk implied by the investor's guarantee and/or the non optimality of his resulting constrained portfolio with regards to his own risk aversion. Finally, we choose the standard OBPI strategy as benchmark to both evaluate the investor's and banker's compensating variations. Finally, Section 6 contains the main conclusions. Some of the proofs and most of the figures are gathered in appendices.

[3] The constraint on the terminal wealth is much more general than previous works about insurance portfolio and so can be applied to all practical cases.

[4] The other cases (logarithm and CARA) can be illustrated as well. However, CRRA utilities generally fit better the true utility. Additionally, as shown for the CRRA case, the numerical values of the compensating variations are sufficiently significant to illustrate them.

2 Individual optimal portfolio profiles

2.1 The financial model

We assume the existence of two basic financial assets: the bond B and the stock S (a financial index such as for example the S&P 500, which is considered as a benchmark)). We suppose that the investor determines an optimal payoff h which is a function defined on all possible values of the assets (B, S) at maturity. If the market is complete, this payoff can be achieved by the investor. The market can be complete for example if the financial market evolves in continuous time and all options can be dynamically duplicated by a perfect hedging strategy. It can be complete if for example, in a one period setting, European options of all strikes are available on the financial market. In this setting, the inability to trade continuously potentially induces investment in cash, asset B, asset S and all European options with underlyings B and S (if cash and bond are non stochastic, only European options on S are required).

The asset prices are calculated under risk neutral probabilities. If markets exist for out-of-the-money European puts and calls of all strikes, then it implies the existence of an unique risk-neutral probability that may be identified from option prices (see Breeden and Litzenberger, 1978). Otherwise, if there is no continuous-time trading, generally the market is incomplete and a one particular risk-neutral probability $\mathbb{Q}$ is used to price the options. It is also possible that stock prices change continuously but the market may be still dynamically incomplete. Again, it is assumed that one risk-neutral probability is selected. Assume that prices are determined under such measure $\mathbb{Q}$. Denote by $\frac{d\mathbb{Q}}{d\mathbb{P}_i}$ the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to the historical probability $\mathbb{P}_i$ corresponding to investor i beliefs. Denote by $M_{i,T}$ the density $\frac{d\mathbb{Q}}{d\mathbb{P}_i}$.

2.2 Spanning

Proposition 1 *The payoff h associated to an investment strategy can be computed by the following approach. As proved in Carr and Madan (2001), it is possible to explicitly identify the position that must be taken in order to achieve a given payoff h that is twice differentiable.h is duplicated by an unique initial position of $h(S_0) - h'(S)S$ unit discount bonds, $h'(S)$ shares and $h(K)dK$ out-of-the-money options of all strikes K:*

$$\begin{aligned} h(S) &= [h(S_0) - h'(S_0)S_0]B_0 + h'(S_0)S \\ &\quad + \int_0^{S_0} h''(K)(K-S)^+ dK + \int_{S_0}^{\infty} h''(K)(S-K)^+ dK. \end{aligned}$$

The initial portfolio value satisfies:

$$\begin{aligned} V_0[h(.)] &= [h(S_0) - h'(S_0)S_0]B_0 + h'(S_0)S_0 \\ &\quad + \int_0^{S_0} h''(K)P_0(K)dK + \int_{S_0}^{\infty} h''(K)C_0(K)dK, \end{aligned}$$

where $P_0(K)$ and $C_0(K)$ denote respectively the initial put and call prices.

We deduce that the initial value is also given by:

$$V_0 [h(.)] = B_0 \int_0^\infty h(K)q(K)dK,$$

where $B_0 q(K)$ corresponds to the state price density.

2.3 The non insured portfolio

Recall the results of Brennan and Solanki (1981) or Carr and Madan (2001). Consider an investor i ($i \in \{1, ..., n\}$) who wants to maximize the expected utility of his random terminal wealth $V_{i,T}$ for a given horizon T, under the probability $\mathbb{P}_i$. This latter one corresponds to his beliefs about risky asset return probability ($\mathbb{P}_i = \mathbb{P}_{i,S_T}$). We denote by $f_i(s)$ its probability density function (pdf).

The investor's initial wealth $V_{i,0}$ is composed of a weight w_i^B invested on the bond and a weight w_i^S invested on the stock. Denote respectively by B_0 and S_0 the initial financial asset values.

The investor's utility function U_i is supposed to be increasing, concave and twice-differentiable. Suppose as in Karatzas, Lehoczky, Sethi and Shreve (1986) that the marginal utility U_i' satisfies:

$$\lim_{0^+} U_i' = +\infty \text{ and } \lim_{+\infty} U_i' = 0.$$

Denote by J_i the inverse of the marginal utility U_i'.

Due to the no-arbitrage condition, the budget constraint corresponds to the following relation :

$$V_{i,0} = e^{-rT}\mathbb{E}_{\mathbb{Q}}[h_i(S_T)] = e^{-rT}\mathbb{E}_{\mathbb{P}_i}[h_i(S_T)M_{i,T}],$$

where $M_{i,T}$ denotes the Radon-Nikodym density of the risk-neutral probability $\mathbb{Q}$ with respect to the statistical probability $\mathbb{P}_i$.

The investor has to solve the following optimization problem:

$$Max_{h_i}\mathbb{E}_{\mathbb{P}_i}[U_i(h_i(S_T)] \text{ under } V_{i,0} = e^{-rT}\mathbb{E}_{\mathbb{P}_i}[h_i(S_T)M_{i,T}]. \quad (1)$$

To simplify the presentation of the main results, we suppose as usual that the function h fulfils:

$$\int_{\mathbb{R}^+} h_i^2(s) f_i(s)(ds) < \infty.$$

It means that $h \in \mathbb{L}^2(\mathbb{R}^+, \mathbb{P}_i(ds))$ which is the set of the measurable functions with squares that are integrable on $\mathbb{R}^+$ with respect to the distribution $\mathbb{P}_i$.

Introduce the functional Φ_{U_i} which is associated to the utility function U_i and defined on the space $\mathbb{L}^2(\mathbb{R}^+, \mathbb{P}_i(ds))$ by:

$$\text{For any } X \in \mathbb{L}^2(\mathbb{R}^+, \mathbb{P}_i(ds)),\ \Phi_{U_i}(X) = \mathbb{E}_{\mathbb{P}_i}[U(X)].$$

Φ_{U_i} is usually called the Nemitski functional associated with U_i (see for example Ekeland and Turnbull (1983) for definition and basic properties).

Proposition 2 *Introduce the conditional expectation of $M_{i,T}$ under the σ-algebra generated by S_T. Denote it by g_i. Assume that g_i is a function defined on the set of the values of S_T and $g \in \mathbb{L}^2(\mathbb{R}^+, \mathbb{P}_i(ds))$. Then, the optimization problem is reduced to:*

$$Max_{h \in \mathbb{L}^2(\mathbb{R}^+, \mathbb{P}_i(ds))} \int_{R^+} [U_i(h_i(s))] f_i(s) ds, \tag{2}$$

$$\textit{under } V_{i,0} = \int_{R^+} h_i(s) g_i(s) f_i(s) ds.$$

We deduce the optimal payoff h_i^* is given by:

$$h_i^* = J(\lambda_i g_i),$$

where λ_i is the scalar Lagrange multiplier such that $V_0 = \int_{\mathbb{R}^+} J_i(\lambda_i g_i(s)) g_i(s) f_i(s) ds$.

Proof. It is similar to the proof in Carr and Madan (2001). From the properties of the utility function U_i, the Nemitski functional Φ_{U_i} is concave and differentiable (the Gâteaux-derivative exists) on $\mathbb{L}^2(\mathbb{R}^+, \mathbb{P}_i(ds))$. Additionally, the budget constraint is a linear function of h_i. Thus, there exists exactly one solution h_i^*. It corresponds to the solution of $\frac{\partial L_i}{\partial h_i^*} = 0$ where the Lagrangian $\mathcal{L}_i$ is defined by:

$$\mathcal{L}_i(h_i, \lambda_i) = \int_{\mathbb{R}^+} [U_i(h_i(s))] f_i(s) ds + \lambda_i \left(V_{i,0} - \int_{\mathbb{R}^+} h_i(s) g_i(s) f_i(s) ds \right). \tag{3}$$

The parameter λ_i is the Lagrange multiplier associated to the budget constraint. Therefore, h_i^* satisfies: $U_i'(h_i^*) = \lambda_i g_i$. Thus, $h_i^* = J_i(\lambda_i g_i)$. ■

2.4 The insured portfolio

This section is a generalization of Prigent (1999) and Bertrand, Lesne and Prigent (2001) to the case of heterogeneous beliefs. Now, the investor introduces a specific guarantee, which can be imposed for example by institutional constraints or if he searches for an additional insurance against risk. Such guarantee can be modelled by letting a function $h_{i,g}$ defined on the possible values of the benchmark S_T: whatever the value of S_T, the investor wants to get a final portfolio value above the floor $h_{i,g}(S_T)$. For instance, if $h_{i,g}$ is linear with $h_{i,g}(s) = \alpha_i s + \beta_i$, then, when the benchmark falls, the investor is sure of getting at least the amount β_i (equal to a fixed percentage of his initial investment) and if the benchmark rises, he can capitalize on the rises at a percentage α_i.

The optimal payoff with insurance constraints on the terminal wealth is solution of the following problem:

$$\begin{aligned} & Max_{h_i} \mathbb{E}_{\mathbb{P}_i}[U_i(h_i(S_T)] \\ \text{with } V_{i,0} &= e^{-rT} \mathbb{E}_{\mathbb{P}_i}[h_i(S_T) M_{i,T}], \\ \text{and } h_i(S_T) &\geq h_{i,g}(S_T). \end{aligned} \tag{4}$$

Proposition 3 *The optimal payoff h_i^{**} can be determined by introducing the unconstrained optimal payoff h_i^* associated to the modified coefficient $\lambda_{i,c}$ (i.e. $h_i^* = J_i(\lambda_c g_i)$). $\lambda_{i,c}$ can also be considered as a Lagrange multiplier associated to a non insured optimal portfolio but with a modified initial wealth. Indeed, when h_i^* is greater than the insurance floor $h_{i,g}$, then $h_i^{**} = h_i^*$. Otherwise, $h_i^* = h_{i,g}$. However, the payoff is usually a continuous function of the values of the benchmark like any linear combination of standard options. In that case, the optimal payoff is given by:*

$$h_i^{**} = Max(h_{i,g}, h_i^*). \tag{5}$$

Corollary 4 *The optimal insured portfolio corresponds to a combination of the guaranteed amount $h_{i,g}$ and of a put written on the optimal non insured portfolio h_i^* with strike $h_{i,g}$:*

$$h_i^{**} = h_{i,g} + Max(h_i^* - h_{i,g}, 0). \tag{6}$$

Note that, if both $h_{i,g}$ and h_i^* are increasing, then the optimal payoff h_i^{**} is also an increasing function of the benchmark.

2.5 Properties of the optimal payoffs

The properties of the optimal payoff h_i^* as function of the benchmark S can be analyzed. Introduce the risk tolerance $T_{o,i}(h_i(s))$ equal to the inverse of the absolute risk-aversion:

$$T_{o,i}(h_i(s)) = -\frac{U_i'(h_i(s))}{U_i''(h_i(s))}. \tag{7}$$

Corollary 5 *h_i^* is an increasing function of the benchmark S_T if and only if the conditional expectation g_i of $\frac{d\mathbb{Q}}{d\mathbb{P}_i}$ under the σ-algebra generated by S_T is a decreasing function of S_T. More precisely: assume that g_i is differentiable. From the optimality conditions, the derivative of the optimal payoff is given by:*

$$h_i^{*\prime}(s) = \left(-\frac{U_i'(h_i(s))}{U_i''(h_i(s))}\right) \times \left(-\frac{g_i(s)'}{g_i(s)}\right) = T_{o,i}(h_i^*(s))\frac{d}{ds}\left(Log\left[\frac{1}{g_i(s)}\right]\right). \tag{8}$$

Proof. Since the utility function U_i is concave, the marginal utility U_i' is decreasing, then J_i also, from which the result is immediately deduced. ■

Remark 6 *In most cases g_i is decreasing.*

As it can be seen, $h_i'(s)$ depends on the risk tolerance. The design of the optimal payoff can also be specified, in particular the concavity/convexity property. For this purpose, we can examine the second-order derivative of the payoff. Denote $Y_i(s) = -\frac{g_i'(s)}{g_i(s)}$. We deduce:

Corollary 7 *Assume that g_i is twice-differentiable. Then:*

$$h_i''(s) = [T_{o,i}'(h(s)) + \frac{Y_i'(s)}{Y_i(s)^2}] \times [T_{o,i}(h_i(s))Y_i^2(s)]. \tag{9}$$

Therefore, usually, the higher the tolerance to risk, the higher the second-order derivative h_i"(s).

2.6 Individual prices

Let K_i be the convex cone corresponding to the insurance constraint $h_i \geq h_{i,0}$. Consider the following indicator function of K_i, denoted by δ_{K_i} and defined by:

$$\delta_{K_i}(h_i) = \begin{cases} 0 & if \quad h_i \in K_i \\ +\infty & if \quad h_i \notin K_i \end{cases} \tag{10}$$

In the presence of insurance constraints, the Lagrangian (3) is given by:

$$\mathcal{L}_i(h_i, \lambda_i) = \int_{\mathbb{R}+} [U_i(h_i(s))] f_i(s) ds + \lambda_i \left(V_{0,i} - \int_{\mathbb{R}+} h_i(s) g_i(s) f_i(s) ds \right) + \delta_{K_i}(h_i),$$

where the parameter λ_i is the Lagrange multiplier associated to the budget constraint.

We deduce that the optimal payoff h_i^{**} is solution of the following equation:

$$\frac{f_i(s)}{B_0 q(S)} U_i' \left[h_i^{**}(s)\right] = \lambda_i + \delta_{K_i} \tag{11}$$

where λ_i is the scalar Lagrange multiplier such that $V_0 = \int_{\mathbb{R}+} J_i(\lambda_i g_i(s)) g_i(s) f_i(s) ds$.

$$\pi_i(s) = \frac{f_i(s) U_i' \left[h_i^{**}(s)\right]}{\int_0^\infty B_0 f_i(s) U_i' \left[h_i^{**}(s)\right]} = q(S).$$

2.7 Basic examples

The previous properties are illustrated in next examples. In what follows, we assume that the interest rate r is constant. The stock price evolves in a continuous-time set up. The risky asset price $(S_t)_t$ follows a geometric Brownian motion under investor i beliefs, which is given by:

$$S_t = S_0 exp\left[(\mu_i - 1/2\sigma_i^2)t + \sigma_i W_t\right]. \tag{12}$$

Notations:

$$\theta_i = \frac{\mu_i - r}{\sigma_i}, \; A_i = -\frac{1}{2}\theta_i^2 T + \frac{\theta_i}{\sigma_i}\left(\mu_i - \frac{1}{2}\sigma_i^2\right) T,$$

$$\psi_i = e^{A_i}(S_0)^{\frac{\theta_i}{\sigma_i}}, \; \kappa_i = \frac{\theta_i}{\sigma_i}.$$

Recall that in the Black and Scholes model, the conditional expectation g_i of $\frac{d\mathbb{Q}}{d\mathbb{P}_i}$ under the σ-algebra generated by S_T is given by:

$$g_i(s) = \psi_i s^{-\kappa_i}.$$

In what follows, we illustrate the results for the base numerical case:

$$\mu = 7\%; r = 3\%, \sigma = 20\%; T = 5 \ years; S_0 = 100; V_0 = 1000; p = 1. \tag{13}$$

We restrict the set of possible utility functions U_i to those which exhibit a linear risk tolerance (LRT).[5]

$$T_{o,i}(v) = -\frac{U_i'}{U_i''}(v) = \tau_i + b_i v, \tag{14}$$

where b_i corresponds to cautiousness assumed to be non-negative. The portfolio value is assumed to satisfy: $v \geq -\frac{\tau_i}{b_i}$ so that the risk tolerance is always positive. As the portfolio value converges to this lower bound, the tolerance tends to 0. Therefore, there exists a floor equal to $-\frac{\tau_i}{b_i}$. As in Carr and Madan (2001), we require $V_{i,0} \geq B_0 \left(-\frac{\tau_i}{b_i}\right)$ which allows the portfolio value to reach this floor.

First, Equation (14) can be solved to determine the marginal utility. Secondly, by integrating, we deduce all possible utility types (up to positive linear transformations).

2.7.1 The CARA case

Assume that the utility function of the investor is a CRRA utility: (it corresponds to $b_i = 0$).

$$U_i(x) = -\frac{\exp\left[-a_i x\right]}{a_i}, x > 0,$$

with $a_i = \frac{1}{\tau_i} > 0$, from which we deduce $J_i(y) = -\frac{\ln[y]}{a_i}$. The parameter a_i corresponds to the constant absolute risk aversion.

By using the previous general results about the optimization problem, we deduce:

1) If there is no insurance constraint, the optimal payoff is given by:

$$h_i^*(s) = J_i(\lambda_i g_i(s)) = -\frac{1}{a_i}\left[\ln(\lambda_i) + \ln(\psi_i)\right] + \frac{\kappa_i}{a_i}\ln(s), \tag{15}$$

where λ_i is the Lagrange parameter linked to the budget constraint.

Substituting the Lagrange parameter, we get:

$$h_i^*(s) = V_{i,0}e^{rT} + \frac{\kappa_i}{a_i}\left[\ln(s) - \int_0^\infty \ln(s) g_i(s) f(s) ds\right]. \tag{16}$$

[5] As mentioned in Carr and Madan (2001), Cass and Stiglitz (1970) have proved that a necessary condition to get the two-fund monetary separation is that investors have linear risk tolerance. See also Gollier (2001) for more details about the choice of the utility function.

2) If the insurance constraint is required then the optimal payoff must be solution of:

$$
\begin{aligned}
& Max_{h_i} \mathbb{E}_{\mathbb{P}_i}\left[-\frac{\exp\left[-a_i h_i(S_T)\right]}{a_i}\right] \\
V_{i,0} \quad &= \quad e^{-rT}\mathbb{E}_{\mathbb{Q}_i}[h_i(S_T)], \\
h_i(S_T) \quad &\geq \quad h_{i,g}(S_T). \qquad (17)
\end{aligned}
$$

Then, we deduce that the optimal payoff with guarantee is given by:

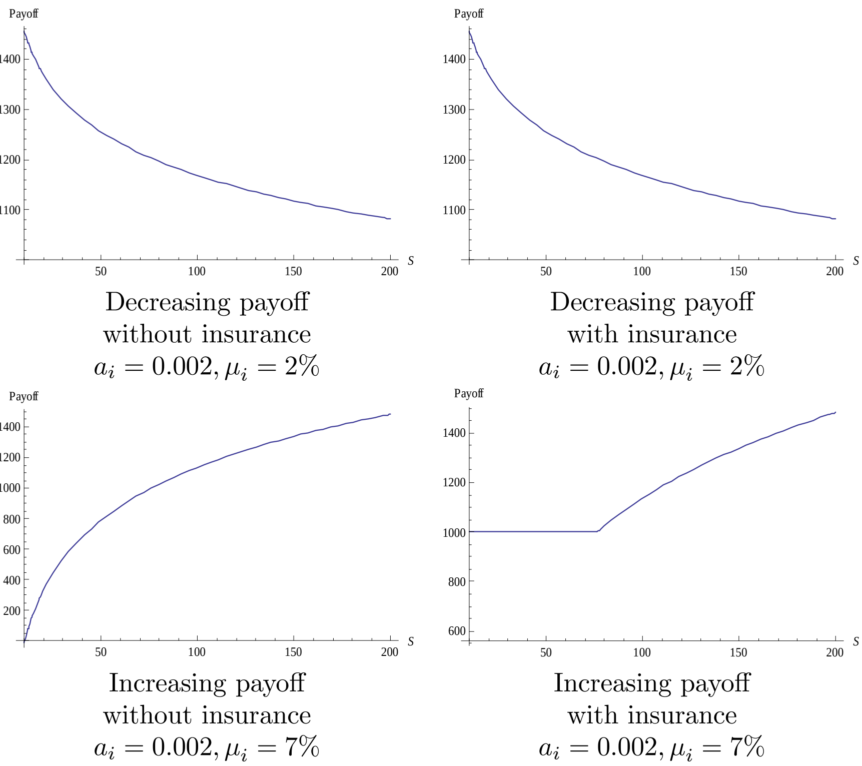
$$h_i^{**} = Max(h_{i,g}, h_i^*), \qquad (18)$$

where h_i^* is given in Relation (15) for an adequate initial investment $\widetilde{V}_{i,0}$.

Thus, we face two main cases:

1) $\mu_i < r$. In that case, the Sharpe type ratio κ_i is negative, which implies that the optimal payoff is decreasing with respect to the risky asset.

2) $\mu_i > r$. In that case, the Sharpe type ratio κ_i is positive, which implies that the optimal payoff is increasing with respect to the risky asset.



Decreasing payoff without insurance $a_i = 0.002, \mu_i = 2\%$

Decreasing payoff with insurance $a_i = 0.002, \mu_i = 2\%$

Increasing payoff without insurance $a_i = 0.002, \mu_i = 7\%$

Increasing payoff with insurance $a_i = 0.002, \mu_i = 7\%$

Figure 1: CARA Profiles

2.7.2 The logarithm case

Assume that the utility function of the investor is a logarithmic utility: (it corresponds to $b_i = 1$)

$$U_i(x) = \ln[\tau_i + x], x > -a_i,$$

with $\tau_i > 0$, from which we deduce $J_i(y) = \frac{1}{y} - \tau_i$.

The previous general results yield to the following result:

1) If there is no insurance constraint, the optimal payoff is given by:

$$h_i^*(s) = \frac{1}{\lambda_i g_i(s)} - \tau_i = \frac{1}{\lambda_i \psi_i} s^{\kappa_i} - \tau_i, \tag{19}$$

where λ_i is the Lagrange parameter linked to the budget constraint.[6]

Substituting the Lagrange parameter, we get:

$$h_i^*(s) = \left(V_{i,0}e^{rT} + \tau_i\right) \frac{1}{\psi_i} s^{\kappa_i} - \tau_i. \tag{20}$$

2) If the insurance constraint is required then the optimal payoff must be solution of:

$$\begin{array}{rcl} & & Max_{h_i} \mathbb{E}_{\mathbb{P}_i}\left[\ln\left[\tau_i + h_i(S_T)\right]\right] \\ V_{i,0} & = & e^{-rT}\mathbb{E}_{\mathbb{Q}_i}[h_i(S_T)], \\ h_i(S_T) & \geq & h_{i,g}(S_T). \end{array} \tag{21}$$

Then, we deduce that the optimal payoff with guarantee is given by:

$$h_i^{**} = Max(h_{i,g}, h_i^*), \tag{22}$$

where h_i^* is given in Relation (23) for an adequate initial investment $\widetilde{V}_{i,0}$.

The optimal payoff is concave or convex according to conditions $\kappa_i < 1$ or $\kappa_i > 1$.

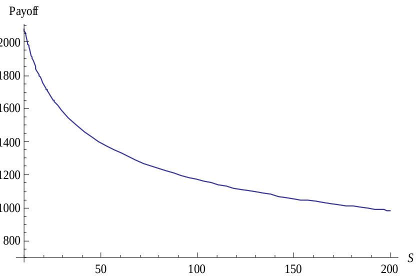
2.7.3 The HARA case

The HARA case without additional guarantee constraint Assume that the utility function of the investor is a HARA utility: (it corresponds to $b_i \neq 0$ and $b_i \neq 1$)

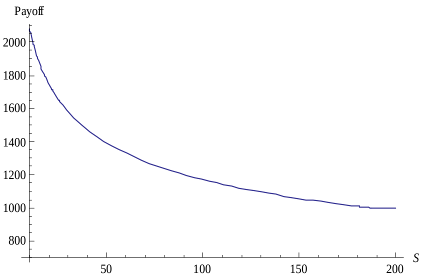
$$U_i(x) = \frac{(x - \widehat{x}_i)^{1-\gamma_i}}{1-\gamma_i}, \; x > \widehat{x}_i,$$

with $\gamma_i = \frac{1}{b_i}$and $\widehat{x}_i = -\frac{\tau_i}{b_i}$.

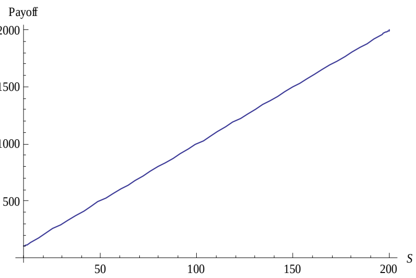
[6] Note that in that case, it would be more convenient to restrict the set of all possibles values of the risky asset.



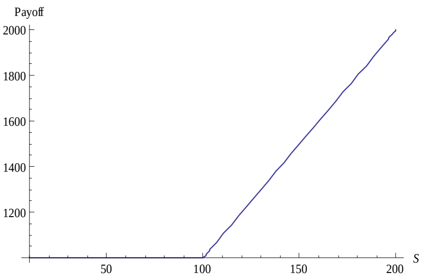
Decreasing payoff
without insurance
$\tau_i = 0.002, \mu_i = 2\%$



Decreasing payoff
with insurance
$\tau_i = 0.002, \mu_i = 2\%$



Increasing payoff
without insurance
$\tau_i = 0.002, \mu_i = 7\%$



Increasing payoff
with insurance
$\tau_i = 0.002, \mu_i = 7\%$

Figure 2: Logarithm Profiles

We deduce: $J_i(y) = \widehat{x}_i + y^{\frac{1}{\gamma_i}}$.
The relative risk aversion is given by:

$$-x\frac{U"_i(x)}{U'_i(x)} = \gamma_i \frac{x}{(x - \widehat{x}_i)}.$$

If $\widehat{x}_i = 0$, the relative risk aversion is equal to the inverse of the risk tolerance: $\gamma_i = \frac{1}{b_i}$.

The CRRA case with an additional guarantee constraint Assume that the utility function of the investor is a CRRA utility. This corresponds to previous HARA case but with $\widehat{x}_i = 0$.

$$U_i(x) = \frac{x^{1-\gamma_i}}{1-\gamma_i}, \; x > 0,$$

from which we deduce $J_i(y) = y^{\frac{1}{\gamma_i}}$.
We apply the previous general results to solve the optimization problem.

Then, if there is no insurance constraint, the optimal payoff is given by:

$$h_i^*(s) = \frac{V_{i,0}e^{rT}}{\int_0^\infty g_i(s)^{\frac{1-\gamma_i}{-\gamma_i}} f_i(s)ds} \times g_i(s)^{\frac{-1}{\gamma_i}}. \tag{23}$$

Therefore, $h_i^*(s)$ satisfies:

$$h_i^*(s) = d_i \times s^{m_i} \text{ with } d_i = c_i\psi_i^{\frac{1}{\gamma_i}} \text{ and } m_i = \frac{\kappa_i}{\gamma_i} > 0. \tag{24}$$

If the insurance constraint is required then the optimal payoff must be solution of:

$$\begin{aligned} & Max_{h_i}\mathbb{E}_{\mathbb{P}_i}\left[\frac{(h_i(S_T))^{1-\gamma_i}}{1-\gamma_i}\right] \\ V_{i,0} &= e^{-rT}\mathbb{E}_{\mathbb{Q}_i}[h_i(S_T)], \\ h_i(S_T) &\geq h_{i,g}(S_T). \end{aligned} \tag{25}$$

Then, we deduce:

Proposition 8 *The optimal payoff with guarantee is given by:*

$$h_i^{**} = Max(h_{i,g}, h_i^*), \tag{26}$$

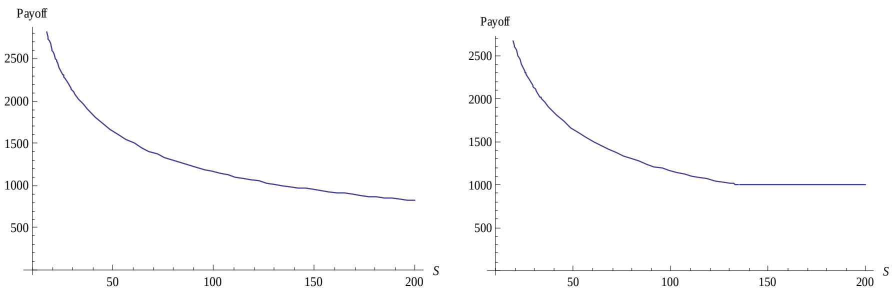
where h_i^* is given in Relation (23) for an adequate initial investment $\widetilde{V}_{i,0}$.

Corollary 9 *Assume as usual that $h_{i,g}$ is increasing and continuous, then the optimal payoff is an increasing continuous function of the benchmark at maturity.*

Corollary 10 *If there is no insurance constraint, the concavity/convexity of the optimal payoff is determined by the comparison between the risk-aversion and the ratio $\kappa_i = \frac{\mu_i - r_i}{\sigma_i^2}$ which is the Sharpe ratio divided by the volatility σ.*
i) h_i^* is concave if $\kappa_i < \gamma_i$.
ii) h_i^* is linear if $\kappa_i = \gamma_i$.
iii) h_i^* is convex if $\kappa_i > \gamma_i$.

Remark 11 *As it can be seen, the graph of the optimal payoff changes from concavity to convexity according to the increase of the risk-aversion of the investor. If for example, the insurance constraint is linear ($h_{i,g}(s) = \alpha_i s + \beta_i$), it looks like the unconstrained case's one, except when h_i^* is equal to the constraint $h_{i,g}$. Consider the dynamically complete case where the stock price follows the usual geometric Brownian motion. If we maximize the expected CRRA utility of the difference between the portfolio value and the floor, we find (see Prigent (2001)) that the CPPI method is optimal with a multiple m_i equal to $\frac{\mu_i - r}{\sigma_i^2}\frac{1}{\gamma_i}$. Again, the concavity/convexity of the optimal payoff depends on the comparison between the Sharpe type ratio $\frac{\mu_i - r}{\sigma_i^2}$ and the risk aversion.*

Decreasing payoff ($\mu_i < r$)



Decreasing payoff without insurance ($\mu_i = 2\%$)

Decreasing payoff with insurance ($\mu_i = 2\%$)

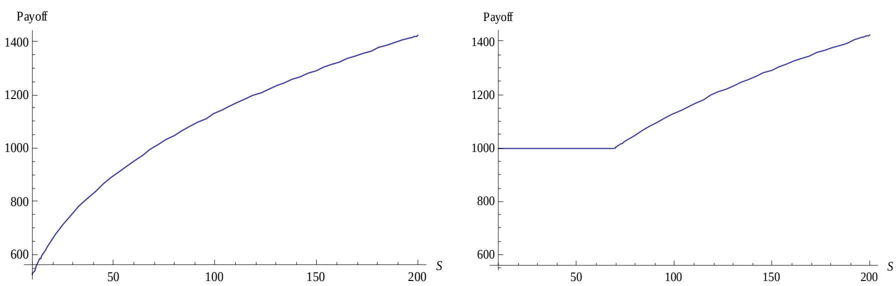
Figure 3: CRRA Decreasing Profiles

Due to bad financial performance, the investor search for high portfolio values for bearish market (for example, for $S_T \leq 0.75\ S_0$, the investor's portfolio return is higher than 35%). For high risky asset returns, he recovers exactly the insured portfolio return, here equal to 1 (for example, for $S_T \geq 1.35\ S_0$).

Increasing payoff ($\mu_i > r$)

Two main cases must be distinguished: the concave case and the convex case.

- If $\kappa_i < \gamma_i$, h_i^{**} is concave.



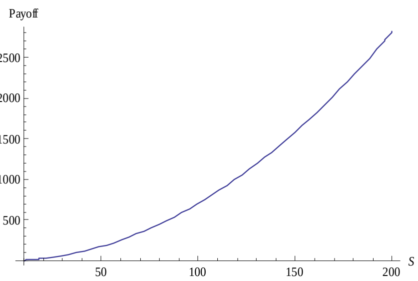
Concave payoff without insurance ($\gamma = 3$)

Concave payoff with insurance ($\gamma = 3$)

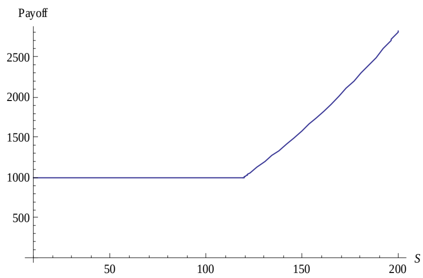
Figure 4: CRRA Concave Profiles

In this case, the investor has a significant relative risk aversion and/or the financial market has not a good performance (its Sharpe type ratio is weak). Therefore, a rather conservative investor searches for not too small returns when the financial market is bearish. But, he has also to require a specific additional guarantee if he wants to recover his initial capital at maturity. However, for moderate bullish market, he does not make high benefits.

- If $\kappa_i > \gamma_i$, h_i^{**} is convex.



Convex payoff
without insurance
($\gamma = 0.5$)



Convex payoff
with insurance
($\gamma = 0.5$)

Figure 5: CRRA Convex Profiles

In that case, the investor has a moderate or weak relative risk aversion and/or the financial market has a good performance (its Sharpe type ratio is relatively high). Thus, a rather aggressive investor searches for the highest possible returns when the financial market is bullish. But, he has also to require a specific additional guarantee if he wants to recover his initial capital at maturity. However, for bearish or moderate bullish market, he receives only this guaranteed amount.

For the special case $\kappa_i = \gamma_i$, the optimal portfolio is linear with respect to the benchmark S. This case corresponds to the standard buy-and-hold strategy. The investor's portfolio does not involve any derivative asset.

3 Optimal insured portfolio profiles in a general equilibrium

In this section, we determine the optimal positioning and the security design when the risk-neutral pricing is endogenous and based on market clearing conditions. We extend previous results of Carr and Madan (2001) by introducing general insurance constraints on the individual portfolios. In the seminal paper of Leland (1980), it is proved that, when the investor has the same market's beliefs, then the optimal payoff is globally convex if and only if the investor's risk tolerance increases with his portfolio value more rapidly than the aggregate investor's risk tolerance increases with global market wealth. Under the assumption of linear risk tolerance with identical cautiousness and log-normal beliefs with same volatility, Leland (1980) proves that the investor's optimal payoff is globally convex if and only if he has a higher expected return than the financial market. Carr and Madan (2001) show for example that, if the investor thinks that the volatility is going to be high, then he will sell at-the-money options while, if he thinks that the volatility is going to be small, then he will purchase out-of-the money options. Benninga and Mayshar (2000) examine the demand for options in a general equilibrium framework by agents having HARA utility functions. They prove that heterogeneity in the degree of relative risk aversion among investors implies that the representative agent has a decreasing relative risk aversion. They show that the Black-Scholes formula does no longer hold and that all options are overpriced with respect to the Black-Scholes prices. Franke *et al.* (2000) prove also that background risk can potentially explain option demand rather than heterogeneity in preferences.

In what follows, we determine the optimal payoffs given the utility functions, the insurance constraints, the bond and asset prices and the probability beliefs. However, the option prices are determined endogenously. The agents must trade so that in particular they exchange their derivatives position. This is the option market clearing position.

Consequently, we have the two following market clearing conditions:

$$\sum_{i=1}^{n} V_{i,0} = S_0 + B_0 \text{ and } \sum_{i=1}^{n} h_i(S) = \alpha S + \beta B, \tag{27}$$

which implies $\sum_{i=1}^{n} h'_i(S) = \alpha$ and zero-net supply on the option markets. Parameters α and β correspond respectively to the shares of the stock market S and the bond market B. As in Carr and Madan (2001), we can set for simplicity: $\alpha = 1$ and $\beta = 1$.[7]

[7] However, when dealing with contract between specific agents such as a financial institution ("the banker") and his customer ("the investor"), we can impose for instance that the initial stock position is null (see next section that deals with compensating variations with possible initial risk-neutral hedging condition imposed to the banker). This point of view is more microeconomics than the assumption $\alpha = \beta = 1$, for which we assume implicitly that we involve all financial isntitutions together with their clients.

From individual expected utility maximization, we can derive the general financial equilibrium properties in this static economy.

3.1 The equilibrium risk-neutral probability

The optimal positioning of a given investor is mainly based on the attributes of this agent. The influence of the other investors is summarized by the risk-neutral density that take account of their preferences and beliefs. In what follows, we consider an economy in which several investors optimize simultaneously their respective portfolio profiles. The option prices are no longer taken as given. It implies that the risk-neutral density $q(.)$ must be determined endogenously.[8] This latter one must satisfy the bond pricing condition:

$$B_0 \int_0^{+\infty} q(s)ds = B_0, \tag{28}$$

which is equivalent to:

$$\int_0^{+\infty} g_i(s)ds = 1 \ (g_i(.) \text{ is a pdf}).$$

We suppose as in Carr and Madan (2001) that bonds and options are in zero net supply. Thus, in the aggregate, only the stock is held:

$$\sum_{i=1}^{n} h_i(S_T) = S_T. \tag{29}$$

The previous conditions imply that the risk-neutral expected return on the stock is equal to the riskless rate. Indeed, since each investor is endowed with δ_i shares with $\sum_{i=1}^{n} \delta_i = 1$ and that $V_{0,i} = \delta_i S_0$, we deduce that:$\sum_{i=1}^{n} V_{0,i} = S_0$.

Using budget conditions, we get:

$$\int_0^{+\infty} B_0 \sum_{i=1}^{n} h_i(s)q(s)ds = S_0,$$

from which we conclude that:

$$B_0 \int_0^{+\infty} sq(s)ds = S_0 \ (\text{no-arbitrage condition}). \tag{30}$$

We take S_0 as given and search for the risk-neutral probability density $q(.)$, the bond price and the optimal portfolio payoffs. In order to determine the

[8] Recall that the risk-neutral density is equal to:

$$q(s) = g_i(s)f_i(s),$$

where $f_i(.)$ is the pdf of the risky asset value S_T under the statistical probability $\mathbb{P}$ and that $g_i(.)$ is the pdf of the Radon-Nikodym density of the risk-neutral probability.

risk-neutral probability density at equilibrium, recall that the optimal portfolio profiles with insurance constraints are given by:

$$h_i^{**}(s) = h_{i,g}(s) + Max\left[h_i^*(s) - h_{i,g}(s), 0\right], \tag{31}$$

where $h_{i,g}$ denotes the guaranteed payoff and $h_i^*(s)$ corresponds to the optimal payoff without insurance constraint (but with a modified initial wealth). From Relation (8), the payoff $h_i^*(s)$ is given by:

$$\frac{d}{ds}\left[h_i^*(s)\right] = T_{i,o}(h_i^*(s))\frac{d}{ds}\ln\left[\frac{f_i(s)}{q(s)}\right]. \tag{32}$$

Thus, assuming that the guaranteed payoff $h_{i,g}$ is differentiable,[9] the derivative of the optimal payoff with insurance constraint exists[10] and is given by:

$$\frac{d}{ds}\left[h_i^{**}(s)\right] = \frac{d}{ds}\left[h_{i,g}(s)\right] \text{ if } h_i^*(s) < h_{i,g}(s), \tag{33}$$

$$\text{and} \tag{34}$$

$$\frac{d}{ds}\left[h_i^{**}(s)\right] = T_{i,o}(h_i^*(s))\frac{d}{ds}\ln\left[\frac{1}{g_i(s)}\right] \text{ if } h_i^*(s) > h_{i,g}(s). \tag{35}$$

Therefore, summing over i, we get:

$$\sum_{i=1}^{n}\frac{d}{ds}\left[h_i^{**}\right](s) =$$

$$\sum_{i=1}^{n}T_{i,o}(h_i^*(s))\left[\frac{d}{ds}\ln\left[f_i(s)\right] - \frac{d}{ds}\ln\left[q(s)\right]\right]\mathbb{I}_{\left\{h_i^*(s)>h_{i,g}(s)\right\}} + \frac{d}{ds}\left[h_{i,g}(s)\right]\mathbb{I}_{\left\{h_i^*(s)<h_{i,g}(s)\right\}}$$

Consequently, since $\sum_{i=1}^{n}\frac{d}{ds}\left[h_i^{**}\right](s) = 1$, we get:

$$\frac{d}{ds}\ln\left[q(s)\right] =$$

$$\frac{-1 + \sum_{i=1}^{n}T_{i,o}(h_i^*(s))\left[\frac{d}{ds}\ln\left[f_i(s)\right]\right]\mathbb{I}_{\left\{h_i^*(s)>h_{i,g}(s)\right\}} + \frac{d}{ds}\left[h_{i,g}(s)\right]\mathbb{I}_{\left\{h_i^*(s)<h_{i,g}(s)\right\}}}{\sum_{i=1}^{n}T_{i,o}(h_i^*(s))\mathbb{I}_{\left\{h_i^*(s)>h_{i,g}(s)\right\}}}. \tag{36}$$

Solving (36), we deduce:

[9] For the standard case, its derivative is nul since it is constant.

[10] There exists only a finite set of values $\widetilde{s}$ at which h_i^{**} is non differentiable. It corresponds to the case: $h_i^*(\widetilde{s}) = h_{i,g}(\widetilde{s})$. Generally, there exists only one such point $\widetilde{s}$. However, under usual assumptions on the existence of a pdf for the risky asset, this point has a nul probability, so that $h_i^{**}(.)$ is differentiable almost everywhere wit respect to S_T.

Proposition 12 *The risk-neutral density q is given by:*

$$\text{For any } s \in \cup_{i=1}^{n} \{h_i^*(s) > h_{i,g}(s)\}, q(s) = q(0) \exp\left[-\int_0^s \frac{dx}{T(x)} \right] \times$$

$$\exp\left[\int_0^s \sum_{i=1}^{n} \left(\frac{T_{i,o}(h_i^*(x))}{T(x)} \left[\frac{f_i'(x)}{f_i(x)} \right] \mathbb{I}_{\{h_i^*(x) > h_{i,g}(x)\}} + \frac{h_{i,g}'(x)}{T(x)} \mathbb{I}_{\{h_i^*(x) < h_{i,g}(x)\}} \right) dx \right] \tag{37}$$

where $T(.)$ is the total risk tolerance for all payoffs higher than the respective guarantees of all investors, that is:

$$T(s) = \sum_{i=1}^{n} T_{i,o}(h_i^*(s)) \mathbb{I}_{\{h_i^*(s) > h_{i,g}(s)\}}. \tag{38}$$

Corollary 13 *When there does not exist any (exogenous) insurance constraint, we get the Carr and Madan (2001) formula:*

$$q(S) = q(0) \exp\left[-\int_0^S \frac{ds}{T_o(s)} \right] \exp\left[\int_0^S \sum_{i=1}^{n} \frac{T_{i,o}(h_i^*(s))}{T_o(s)} \frac{f_i'(s)}{f_i(s)} ds \right], \tag{39}$$

where $T_o(s) = \sum_{i=1}^{n} T_{i,o}(h_i(s))$ denotes the sum of all individual tolerances.

Additionally, for homogeneous beliefs, the risk-neutral density price is given by:

$$q(S) = q(0) \exp\left[-\int_0^S \frac{ds}{T_o(s)} \right] \exp\left[\int_0^S \frac{f'(s)}{f(s)} ds \right]. \tag{40}$$

If we consider the standard guarantee case for which the insured payoff corresponds to a constant percentage of the initial wealth $p_i V_{i,0}$, then we get:

Corollary 14 *If at least one of the insurance constraint is not binding ($s \in \cup_{i=1}^{n} \{h_i^*(s) > p_i V_{i,0}\}$) then*

$$q(s) = q(0) \exp\left[-\int_0^s \frac{dx}{T(x)} \right] \exp\left[\int_0^s \sum_{i=1}^{n} \left(\frac{T_{i,o}(h_i^*(x))}{T(x)} \left[\frac{f_i'(x)}{f_i(x)} \right] \mathbb{I}_{\{h_i^*(x) > p_i V_{i,0}\}} \right) dx \right] \tag{41}$$

Since the tolerance is defined on optimal payoffs that depend themselves on the risk-neutral density q, the two previous relations do not provide explicit expression for q.

Remark 15 *Note that, Relation (38) shows that the equilibrium risk-neutral density is equal to the product of a factor corresponding to the total risk tolerance (i.e.* $\exp\left[-\int_0^S \frac{ds}{T_o(s)}\right]$*), and a factor reflecting the personal beliefs (called "the market" view in Carr and Madan, 2001). Note that the first factor is a positive decreasing function of the risky asset value* S_T *which may induce a change in the mean and add negative skewness. Carr and Madan (2001) shows that, if the pdf* $f(s)$ *is Lognormal and the risk tolerance is constant, then the risk-neutral density* $q(s)$ *does no longer belong to the family of Lognormal distributions but is skewed to the left and have fatter left tails.*

Remark 16 *With (exogenous) insurance constraints, the risk-neutral density is defined on* $\cup_{i=1}^n \{h_i^*(s) > h_{i,g}(s)\}$*. This set of risky asset values* S_T *corresponds to the case for which at least one of the investors payoffs is above the corresponding guaranteed level. Otherwise, all investors recover only their guarantee. Therefore, on* $\cap_{i=1}^n \{h_i^*(s) < h_{i,g}(s)\}$*, due to market clearing conditions, we must have* $\sum_{i=1}^n h_{i,g}(s) = s$*. Thus, for example for fixed guaranteed amounts corresponding to the respective insured proportions of initial wealths, we must assume that* $\sum_{i=1}^n h_i(S_T) = S_T + B_T$*. It means that the bond clearing condition must be relaxed. Otherwise, one of the investor* j *must have a payoff such that* $h_{j,g}(s) = s - \sum_{i=1,\neq j}^n h_{i,g}(s)$*, but such payoff does not generally correspond to a true guarantee. Agent* j *bears the risk induced by the other investors. Note also that if one agent* j *has no specific guarantee (i.e.* $h_{j,g}(s) = -\infty$*) then* $\cup_{i=1}^n \{h_i^*(s) > h_{i,g}(s)\} = \mathbb{R}^+$*. Thus, Relation (37) allows to define the risk-neutral density* q *for all risky asset values.* [11]

Remark 17 *To compare the risk-neutral densities for the non-insurance and insurance cases, we note that on the domain* $\cap_{i=1}^n \{h_i^*(s) > h_{i,g}(s)\}$*, the risk-neutral density has exactly the same form as in the no-insurance case. They only differ by the levels of initial wealths. Indeed, for the insurance case, the initial values of the payoffs* $h_i^*(s)$ *are smaller than the corresponding ones for the no-insurance case. Thus, according to the monotonicity of the ratio* $\frac{T_{i,o}(h_i^*(s))}{T_o(s)}$ *with respect to the initial investment, we can deduce if for example, the risk-neutral density for the insurance case is higher or smaller than the risk-neutral density for the no-insurance case. Note that, when the risk tolerances are constant (CARA case), they are equal on the domain* $\cap_{i=1}^n \{h_i^*(s) > h_{i,g}(s)\}$*, which corresponds usually to rises of the risky asset.*

[11] See next section about the notion of compensating variations, in which we assume for example that one agent ("the banker") bears this risk. Thus, we relax the bond clearing condition to avoid potential too high losses.

3.2 The optimal portfolio profiles at the equilibrium

Substituting Relation (37) into Relation (34), we deduce:

Proposition 18 *The optimal portfolio profiles at the equilibrium are given by:*

$$\frac{d}{ds}\left[h_i^{**}(s)\right] =$$

$$\text{if } h_i^*(s) < h_{i,g}(s), \frac{d}{ds}\left[h_{i,g}(s)\right],$$

and, if $h_i^*(s) > h_{i,g}(s)$,

$$\frac{T_{i,o}(h_i^*(s))}{T(s)} + T_{i,o}(h_i^*(s)) \times \tag{42}$$

$$\left[\frac{d\ln f_i(s)}{ds} - \sum_{j=1}^{n}\left(\frac{T_{i,o}(h_j^*(s))}{T(s)}\frac{d\ln f_i(s)}{ds}\mathbb{I}_{\left\{h_j^*(s)>h_{j,g}(s)\right\}} + \frac{h'_{j,g}(s)}{T(s)}\mathbb{I}_{\left\{h_j^*(s)<h_{j,g}(s)\right\}}\right)\right]$$

For the non-insurance case, we recover the Carr and Madan (2001) formula:

$$h_i^*(s) = \frac{T_{i,o}(h_i(s))}{T(s)} + T_{i,o}(h_i(s)) \times \left(\frac{d\ln f_i(s)}{ds} - \sum_{i=1}^{n}\frac{T_{i,o}(h_i(s))}{T_o(s)}\frac{d\ln f_i(s)}{ds}\right). \tag{43}$$

The first term is due to the investor's risk tolerance relative to all the investors. The second term involves the own investor's risk tolerance and the difference between the investor's belief and a risk tolerance weighted average of the beliefs of other investors in the financial market.

Consider now the case of homogeneous beliefs ($f_i(s) = f(s)$). We get:

Corollary 19 *For homogeneous beliefs, we get: For* $h_i^*(s) > h_{i,g}(s)$,

$$\frac{d}{ds}\left[h_i^{**}(s)\right] = \frac{T_{i,o}(h_i(s))}{T(s)}. \tag{44}$$

Consider two investors with linear risk tolerances and with homogeneous beliefs. Suppose that they have different cautiousness (otherwise, they will not hold derivatives).

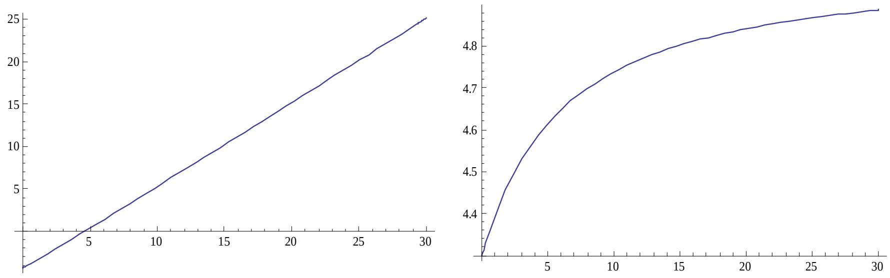
Example 1: Suppose that the two investors have opposite cautiousness:

$$T_{i,1}(v) = \tau_1 + bv, \text{ and } T_{i,2}(v) = \tau_2 - bv.$$

Then, without insurance constraints, we get:

$$\begin{aligned} h_1^*(s) &= \frac{s}{2} - \frac{\tau}{2b} + \sqrt{\left(\frac{s}{2} + \frac{\tau_2 - \tau_1}{2b}\right) + c^2}, \\ h_2^*(s) &= \frac{s}{2} - \frac{\tau}{2b} - \sqrt{\left(\frac{s}{2} + \frac{\tau_2 - \tau_1}{2b}\right) + c^2}, \end{aligned}$$

where $\tau = \tau_1 + \tau_2$ and c is an arbitrary constant. The corresponding portfolio payoffs are displayed in next figure.



Investor 1's portfolio profile   Investor 2's portfolio profile

Figure 6: Investor profiles

When there exists insurance constraints, we get:

Corollary 20 *If there exist only two investors with linear risk tolerances, homogeneous beliefs and opposite cautiousness, then their optimal payoffs under insurance constraints are given by:*

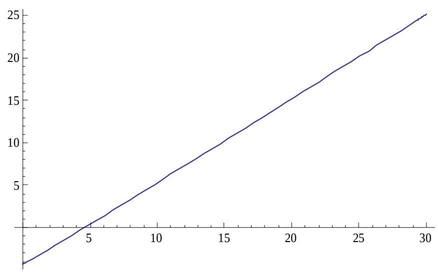
(1) If $h_1^{**}(s) > h_{1,g}(s)$ and if $h_2^{**}(s) > h_{2,g}(s)$,

$$\begin{aligned} h_1^{**}(s) &= \frac{s}{2} - \frac{\tau}{2b} + \sqrt{\left(\frac{s}{2} + \frac{\tau_2 - \tau_1}{2b}\right) + c^2}, \\ h_2^{**}(s) &= \frac{s}{2} - \frac{\tau}{2b} - \sqrt{\left(\frac{s}{2} + \frac{\tau_2 - \tau_1}{2b}\right) + c^2}. \end{aligned} \tag{45}$$

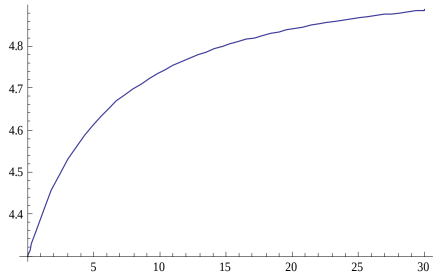
(2) If $h_1^{**}(s) < h_{1,g}(s)$ or if $h_2^{**}(s) < h_{2,g}(s)$, we use the clearing relation to deduce the payoffs.

Proof. See Appendix. ■

Next figure illustrates this case for guarantees corresponding to fixed percentages of the initial invested amounts.



Investor 1's portfolio profile



Investor 2's portfolio profile

Figure 7: Investor "insured" profiles

Example 2: Suppose that the two investors have Lognormal beliefs and negative exponential utility functions:

$$T_{i,1}(v) = \tau_1, \text{ and } T_{i,2}(v) = \tau_2.$$

The optimal payoffs of investors are the solutions of:

$$h_i'^{**}(s) = \frac{\tau_i}{\tau} + \tau_i \left[\frac{d \ln f_i(s)}{ds} - \sum_{i=1}^{n} \frac{\tau_i}{\tau} \frac{d \ln f_i(s)}{ds} \right] \text{ if } h_i^*(s) > h_{i,g}(s),$$

where $\tau = \tau_1 \mathbb{I}_{\left\{h_1^*(s) > h_{1,g}(s)\right\}} + \tau_2 \mathbb{I}_{\left\{h_2^*(s) > h_{2,g}(s)\right\}}$.

Then, we get: if $h_i^*(s) > h_{i,g}(s)$,

$$h_i^{**}(s) = \chi_i + \frac{\tau_i}{\tau} s + \tau_i \varphi_i(s),$$

with $\varphi_i(s) = \ln f_i(s) - \sum_{i=1}^{n} \left(\frac{\tau_i}{\tau} \ln f_i(s) \right)$ and χ_i is a constant deduced from the budget constraint. We note that, for this particular case, the investor's stock and derivatives positions do not depend on his initial wealth if there exists no exogenous insurance constraint. When such constraint holds, the aggregate risk tolerance depends on comparisons between $h_i^*(s)$ and $h_{i,g}(s)$. This latter one is usually a function of the initial wealth (typically, a given percentage of it). The bond position allows to finance the other ones. Note that the higher the risk tolerance, the higher the shares invested on the stock and options. Note that due to indicator functions $\mathbb{I}_{\left\{h_1^*(s) > h_{1,g}(s)\right\}}$ and $\tau_2 \mathbb{I}_{\left\{h_2^*(s) > h_{2,g}(s)\right\}}$, the aggregate risk tolerance τ is smaller than for the non insurance case. However, on the region $\cap_{i=1}^{2} \left\{h_i^*(s) > h_{i,g}(s)\right\}$, they are equal. Note also, under constant risk tolerance, we get:

$$q(s) = v \exp\left[-s/\tau\right] \prod_{i=1}^{n} \left[f_i(s)\right]^{\frac{\tau_i}{\tau}} \text{ with } v \text{ constant.} \tag{46}$$

Thus, as mentioned in Carr and Madan (2001), the market view consists in a weighted geometric average of the individual densities.

4 Compensating Variations

In this section, we introduce the quantitative index of investor's satisfaction introduced by de Palma and Prigent (2008, 2009) to measure the utility loss when the optimal portfolio is not available on the market. It is based on the standard economic concept of *compensating variation.* If an investor with risk aversion γ and initial investment V_0 can buy his optimal portfolio, his expected utility is $\mathbb{E}[U_\gamma(V_T^*); V_0]$. If this investor selects an optimal portfolio among only those available, then he will get the expected utility $\mathbb{E}[U_\gamma(V_T^{*(\lambda)}); V_0]$. He will get the same expected utility provided that he invests an initial amount $\tilde{V}_0 \geq V_0$. Therefore, this investor requires a compensation $\tilde{V}_0$ which satisfies:

$$\mathbb{E}[U_\gamma(V_T^*); V_0] = \mathbb{E}[U_\gamma(V_T^{*(\lambda)}); \tilde{V}_0]. \tag{47}$$

The amount $\tilde{V}_0$ is in line with the certainty equivalent concept in expected utility analysis. It can be viewed as an "implied initial investment" necessary to maintain the level of expected utility. The same analysis can be applied to the banker. We illustrate numerically the theoretical solutions for banker and investor having both CRRA utilities.[12] In what follows, we examine three main cases: The first one corresponds to an investor who has no direct access to the financial derivatives market. Due to the suboptimality of his standard buy-and-hold portfolio, he may be ready to bear (theoretically) an additional cost to can include options in his portfolio. The second case deals with the banker's compensating variation due to riskier financial position in the presence of the investor's guarantee and/or the bad fit of his resulting constrained portfolio to his own risk aversion. Finally, we take the standard OBPI strategy as benchmark to both evaluate the investor's and banker's compensating variations.

4.1 Compensating variation of buy-and-hold strategy w.r.t. optimal portfolio with derivatives

In this subsection, we consider an investor who can only use a buy-an-hold strategy if he has no access to the financial derivatives market. We determine the compensating variation with respect to the true optimal payoff with derivatives provided by the bank. The level of the compensating variation provides a measure of the monetary loss, due to this friction. Conversely, they allow to measure the interest of the investor to use financial intermediates to manage their portfolios, which provides values of the cost they may accept to benefit from such financial service.[13]

[12] The other cases (logarithm and CARA) can be illustrated as well. However, CRRA utilities generally fit better the true utility. Additionnally, as shown for the CRRA case, the numerical values of the compensating variations are sufficiently significant to illustrate them.

[13] We assume that the investor cannot get a portfolio payoff which involves derivatives covered by a dynamic strategy in continuous-time. Thus, if he wants to include options in his portfolio, he must either buy them on the financial market (with a given additional cost), either buy part of a structured fund managed by a financial institution (assumed here to be a bank for simplicity).

4.1.1 The optimal buy-and-hold strategy

The investor i maximizes his expected utility:

$$Max_{w_S}\mathbb{E}\left[U\left[V_T\right]\right],$$

where V_T denotes the portfolio value at maturity T and w_S corresponds to the proportion of wealth invested on the risky asset S.

The buy-an-hold condition consists in fixing shares during the management period. Therefore, we have:

$$V_T = V_0 \times \left(e^{rT} + w_S(e^{X_T} - e^{rT})\right)$$

The first-order condition implies:

$$\mathbb{E}\left[U'(V_T)(e^{X_T} - e^{rT})\right] = 0,$$

which is equivalent to:

$$\int U'\left[V_0 \times \left(e^{rT} + w_S^*(e^x - e^{rT})\right)\right](e^x - e^{rT})f_X(x)dx = 0, \tag{48}$$

where f_X denotes the pdf of the random variable $X = (\mu_i - 1/2\sigma_i^2)T + \sigma_i W_T$.

If the investor has not access to derivatives, then his utility level is smaller than when the bank provides his true optimal portfolio with derivatives. In what follows, we illustrate the compensating variation corresponding to this particular case.

4.1.2 The compensating variation for the buy-and-hold strategy

Our numerical base case corresponds to the following financial parameters:

$$r = 3\%; \mu = 7\%; \sigma = 20\%; B_0 = 1; S_0 = 100; T = 5; V_{i,0} = 1000; p_i = 1 \tag{49}$$

CRRA case with guarantee.

In this framework, Equation (47) is equivalent to:

$$\mathbb{E}[U_\gamma(p_i V_{i,0} + Max\left[\alpha_i V_{i,0} S_T^{m_i} - p_i V_{i,0}, 0\right])] = \mathbb{E}[U_\gamma(\tilde{V}_{i,0} \times \left(e^{rT} + w_{i,S}^*(e^x - e^{rT})\right))], \tag{50}$$

where α_i is determined from the budget equation:

$$\mathbb{E}_{\mathbb{Q}}\left(p_i V_{i,0} + Max\left[\alpha_i V_{i,0} S_T^{m_i} - p_i V_{i,0}, 0\right]\right) = V_{i,0}e^{rT}$$

Proposition 21 *Since here $U_\gamma(v) = \frac{v^{1-\gamma}}{1-\gamma}$, we deduce that the CV equal to the ratio $\tilde{V}_{i,0}/V_{i,0}$ is given by:*

$$\frac{\tilde{V}_{i,0}}{V_{i,0}} = \left(\frac{\mathbb{E}_{\mathbb{P}}\left[(p_i + Max\left[\alpha_i S_T^{m_i} - p_i, 0\right])^{(1-\gamma_i)}\right]}{\mathbb{E}_{\mathbb{P}}[(e^{rT} + w_{i,S}^*\left[e^x - e^{rT}\right])^{(1-\gamma_i)}]}\right)^{\left(\frac{1}{1-\gamma_i}\right)}$$

The numerical values of the CV $\frac{\tilde{V}_{i,0}}{V_{i,0}}$ are displayed in next Table. We consider both three values for the drift μ and for the volatility σ. Five levels of relative risk aversion (RRA) are introduced: $\gamma = 0.5$ and 2 ("aggressive investors"); $\gamma = 5$ (moderate investor); $\gamma = 7$ and $\gamma = 10$ (more conservative investors). The parameter w_S denotes the optimal weights invested on the risky asset for the buy-and-hold strategy without insurance constraint. The parameter w_S^c denotes the optimal weights invested on the risky asset for the buy-and-hold strategy with insurance constraint (the upper bound to guarantee the capital at maturity corresponds to $1 - p_i e^{-rT}$ with the insured proportion p_i equal to 1). The compensating variation (CV) is expressed in percentage of the true initial wealth invested $V_{i,0}$ on the financial market.

Table 1: Compensating Variations for the CRRA case, w.r.t. the standard buy-and-hold portfolio

RRA γ	$\mu = 5\%$ and $\sigma = 20\%$			$\mu = 7\%$ and $\sigma = 20\%$			$\mu = 10\%$ and $\sigma = 20\%$		
	w_S	w_S^c	CV	w_S	w_S^c	CV	w_S	w_S^c	CV
0.5	100%	14%	6.45%	100%	14%	6.42%	100%	14%	15.8%
2	55%	14%	2.37%	80%	14%	2.37%	100%	14%	6.18%
5	0%	0%	2%	30%	14%	1.01%	45%	14%	2.24%
7	0%	0%	1.56%	10%	10%	1.16%	33%	14%	1.39%
10	0%	0%	0.59%	0%	0%	1.34%	0%	0%	2.88%
RRA γ	$\mu = 7\%$ and $\sigma = 15\%$			$\mu = 7\%$ and $\sigma = 20\%$			$\mu = 7\%$ and $\sigma = 25\%$		
	w_S	w_S^c	CV	w_S	w_S^c	CV	w_S	w_S^c	CV
0.5	100%	14%	15.71%	100%	14%	6.41%	100%	14%	5.07%
2	100%	14%	6.18%	80%	14%	2.37%	50%	14%	2.43%
5	45%	14%	2.24%	30%	14%	1.01%	0%	0%	2%
7	34%	14%	1.38%	10%	10%	1.16%	0%	0%	1.73%
10	0%	0%	2.92%	0%	0%	1.34%	0%	0%	1.55%

From Table (1), we note that the compensating variation can be relatively high (up to 15.8%). It is not a monotonous function with respect to financial parameters μ and σ. However, for weak relative risk aversion ($\gamma = 0.5$ or 2), the values of compensating variation are higher than 2.37%. This due to the fact that these investors are rather agressive once the guarantee is fixed. Therefore, they suffer from not having a strictly convex portfolio profile when using a buy-and-hold strategy. For conservative investors ($\gamma = 7$ or 10), the utility loss is less severe since they do not try to make the best benefit from the performance of the risky asset. For a moderate investor ($\gamma = 5$), the compensating variation is about 2%.

4.2 Compensating variations with constraints on the banker

In what follows, we assume that the banker provides exactly the optimal payoff with insurance constraint of the customer. Therefore, the banker may bear a utility loss for not having his own optimal portfolio profile. Indeed, his portfolio profile h_b is determined from Relation (27) based on derivative market clearing conditions. Therefore, we have:

$$h_{b,T} = q_S S_T + q_B B_T - h_{i,T}, \tag{51}$$

where q_S and q_B are the shares respectively invested by the banker on the risky asset S and the riskless asset B, on one hand to partially hedge the payoff h_i sold to the investor and, on the other hand, to partially optimize the expected utility of his financial position h_b.

The parameters q_S and q_B are linked also to the budget constraint of the banker: $V_{b,0} = q_S S_0 + q_B B_0 - V_{i,0}$ and his regulatory risk constraint:

$$\mathbb{P}\left[V_{b,T} - V_{b,0} + VaR\left(\varepsilon\right) \leq 0\right] \leq \varepsilon, \tag{52}$$

where $VaR\left(\varepsilon\right)$ denotes the Value-at-Risk of the financial position at the given probability level ε.

Additional criteria can be introduced to determine the shares q_S and q_B:

Criterion 1: The banker tries to minimize the ratio of the reserve amount upon the initial investment necessary to prevent losses at the given probability level ε.

Criterion 2: ("delta-neutrality") The portfolio manager (the banker) can try to hedge the investor's portfolio profile by setting:

$$q_S = \frac{\partial h_{i,0}}{\partial S_0}. \tag{53}$$

It means that the banker searches for a riskless portfolio but can only hedge the investor's risky position by a static position. Therefore, this leads only to a partial hedging with a residual risk, according to risky asset fluctuations.

4.2.1 Compensating variations with VaR constraints on the banker

The VaR condition can be detailed as follows:

$$\mathbb{P}\left[q_S S_T + q_B B_T - h_{i,T} - V_{b,0} + VaR\left(\varepsilon\right) \leq 0\right] \leq \varepsilon \tag{54}$$

is equivalent to: (recall that $q_S S_0 + q_B B_0 - V_{i,0} = V_{b,0}$, thus $q_B = (V_{i,0} + V_{b,0} - q_S S_0)/B_0$).

$$\mathbb{P}\left[q_S\left(S_T - S_0 e^{rT}\right) - h_{i,T} + e^{rT} V_{i,0} + \left(e^{rT} + \delta - 1\right) V_{b,0} \leq 0\right] \leq \varepsilon,$$

with $VaR\left(\varepsilon\right)/V_{b,0} = \delta$.

Set:

$$c = -q_S S_0 e^{rT} + \left(e^{rT} - p_i\right) V_{i,0} + \left(e^{rT} + \delta - 1\right) V_{b,0}.$$

Then, the VaR condition is equivalent to:

$$\mathbb{P}\left[q_S S_T + V_{i,0} Max(\alpha_i S_T^{m_i} - p_i, 0) + c \leq 0\right] \leq \varepsilon. \tag{55}$$

The previous probability (55) is given by:

$$\mathbb{E}\left[\mathbb{I}_{q_S s + V_{i,0} Max(\alpha_i s^{m_i} - p_i, 0) + c \leq 0}\right].$$

In this framework, the characterization of the banker's compensating variation CV_b is determined from the equality:

$$\mathbb{E}[U_b(h^*_{b,T}); V_{b,0})] = \mathbb{E}[U_b(h_{b,T}); \tilde{V}_{b,0}], \tag{56}$$

where $h^*_{b,T}$ denotes the true optimal payoff for the initial invested amount $V_{b,0}$.

Assume that both the banker and the investor have CRRA utilities with respective RRA γ_b and γ_i. We consider both three values for the drift μ ($\mu = 4\%; \mu = 7\%; \mu = 15\%$) and for the volatility σ ($\sigma = 10\%; \sigma = 20\%; \sigma = 30\%$). Five levels of the investor's relative risk aversion (RRA) are introduced: $\gamma_i = 0.5$ and 2 ("aggressive investors"); $\gamma_i = 5$ (moderate investor); $\gamma_i = 7$ and 10 (more conservative investors). For the banker, we assume the following relative risk aversions: $\gamma_b = 0.1$, 0.5, 2, 5 and 10. We choose the standard probability level: $\varepsilon = 1\%$. For the ratio , we set $\delta = 5\%$.[14]

In Appendix 4, figures about both investor's and banker's payoffs are displayed to illustrate the impact of expected risky asset return μ and volatility σ on the portfolio profiles. We note for example that for high values of μ or weak values of σ (thus high values of Sharpe ratio, which implies rather good performance of the financial market), the constrained banker's payoff can be, on one hand negative for risky asset values far from the spot value S_0, and, on the other hand, decreasing when the risky asset price rises significantly (see Figures A.4.3 and A.4.4). On the contrary, for small risky asset prices or high volatility values (thus small values of Sharpe ratio, which implies weak performance of the financial market), the constrained banker's payoff is most of the time increasing for risky asset values far from the spot value S_0 (see Figures A.4.2 and A.4.5).

Due to Relation (51), the expected utility of the banker when accepting to sell the investor's profile is given by:

$$\frac{1}{1-\gamma_b}\mathbb{E}\left[\left(q_S S_T + q_B B_T - \left(p_i + Max\left[\alpha_i S_T^{m_i} - p_i, 0\right]\right) V_{i,0}\right)^{1-\gamma_b}\right].$$

His expected utility for his best strategy is given by:

$$\frac{1}{1-\gamma_b}\mathbb{E}\left[\left(\alpha_b \times S_T^{m_b}\right)^{1-\gamma_b}\right].$$

To compensate (theoretically) the non optimality of first portfolio, the banker must invest a higher initial amount $\tilde{V}_{b,0}$. Therefore, we deduce:

[14] If $Min_{q_S,q_B} VaR(\varepsilon)/V_{b,0} \geq \delta$, we choose the pair (q_S, q_B) that minimizes $VaR(\varepsilon)/V_{b,0}$.

Proposition 22 *The CV equal to the ratio $\tilde{V}_{b,0}/V_{b,0}$ is given by:*

$$\frac{\tilde{V}_{b,0}}{V_{b,0}} = \left(\frac{\mathbb{E}\left[\left(\alpha_b \times S_T^{m_b} \right)^{1-\gamma_b} \right]}{\mathbb{E}\left[\left(\frac{q_S S_T + q_B B_T - \left(p_i + Max\left[\alpha_i S_T^{m_i} - p_i, 0 \right] \right) V_{i,0}}{\tilde{V}_{b,0}} \right)^{1-\gamma_b} \right]} \right)^{\left(\frac{1}{1-\gamma_b} \right)}.$$

The numerical values of the CV $\frac{\tilde{V}_{b,0}}{V_{b,0}}$ are displayed in Table (2), which provides the CV values for the VaR criterion. We set $V_{0,b} = 1000$.[15] For the banker, we assume the following relative risk aversions: $\gamma_b = 0.1$, 0.5, 2, 5 and 10. We can see that for high Sharpe type ratio due to high return ($\frac{\mu - r}{\sigma^2} = 3$ for the case $\mu = 15\%$ and $\sigma = 20\%$), the (theoretical) CV is very high (except for the less risky position corresponding to $\gamma_i = 10$ and $\gamma_b = 10$). This is mainly due to the high risk of the constrained banker's payoff, due to the convexity of the investors' payoff and/or his additional guarantee demand. Indeed, in this case, the static VaR hedge is inefficient (recall that $T = 5$ years) and the banker's risk is too high, which implies very high CV to theoretically compensate this risk level. For high Sharpe type ratio due to small volatility ($\frac{\mu - r}{\sigma^2} = 4$ for the case $\mu = 7\%$ and $\sigma = 10\%$), the CV lies between 0.2% and 10%. For $\gamma_b = 10$, the CV value is about 2%. These CV values are much smaller than previous ones, since the volatility risk is small. For more standard Sharpe type ratio ($\frac{\mu - r}{\sigma^2} = 1$ for the case $\mu = 7\%$ and $\sigma = 20\%$), the (theoretical) CV can be very high for weak RRA γ_b of the banker ($\gamma_b = 0.1; 0.5$) and/or for weak RRA γ_i of the investor. However, if we assume that the banker does not search for high returns from his investment $V_{b,0}$ but rather for total returns that are almost riskless, then his RRA γ_b must be relatively high, for instance $\gamma_b = 10$. In that case, his CV lies between 4.7% and 7.5%, except for the riskier case ($\gamma_i = 0.5$). For relatively small Sharpe type ratio ($\frac{\mu - r}{\sigma^2} = \frac{4}{9}$, for the case $\mu = 7\%$ and $\sigma = 30\%$), the CV is still high (about 23%) since the volatility is also very significant ($\sigma = 30\%$). Finally, for small Sharpe type ratio ($\frac{\mu - r}{\sigma^2} = \frac{1}{4}$, for the case $\mu = 4\%$ and $\sigma = 20\%$), the CV value lies mainly between 4% and 9%. This is due in particular to the bad performance of the financial market for the investor.

[15] It corresponds to the amount invested by the investor and allow to partially hedge the risk due to the investor's profile. Note that other banker's amount (for instance twice) do not change substantially the main conclusion of this study.

Table 2: CV CRRA (banker) VaR constraint

RRA	$\mu=4\%$ and $\sigma=20\%$			$\mu=7\%$ and $\sigma=20\%$			$\mu=15\%$ and $\sigma=20\%$		
γ_i/γ_b	2	5	10	2	5	10	2	5	10
0.5	4.42%	4.85%	6%	47%	32%	20%	×65	×20	×54
2	4.88%	6.98%	9.83%	6.5%	4.6%	4.7%	×28	×10	×20
5	4.47%	6.32%	8.97%	5.2%	5.3%	5.4%	×1.36	13%	6.5%
10	4.33%	6.09%	8.54%	4.8%	4.9%	7.5%	×1.26	7.5%	5%
RRA	$\mu=7\%$ and $\sigma=10\%$			$\mu=7\%$ and $\sigma=20\%$			$\mu=7\%$ and $\sigma=30\%$		
γ_i/γ_b	2	5	10	0.1	0.5		2	5	10
0.5	7.72%	7.75%	8.2%	83%	85%		26%	28%	34%
2	10.6%	9.5%	9.1%	69%	19.7%		23%	24.5%	27%
5	26%	13%	9.8%	64%	16%		21.8%	23.7%	26.7%
10	8.8%	0.22%	1.84%	64%	16%		21.3%	23%	25.8%

4.2.2 Compensating variations with risk-neutral hedge constraints on the banker

Assume the delta-neutrality characterized by the condition: $q_S = \frac{\partial V_{i,0}}{\partial S_0}$. Here, since we have $V_{i,0} = \left(p_i e^{-rT} V_{i,0} + C_i\right)$ where denotes the initial value of the power call $Max\left[\alpha_i V_{i,0} S_T^{m_i} - p_i V_{i,0}, 0\right]$.

Lemma 23 *Assume that S is the geometric Brownian motion given in (12). Then the share $q_S = \frac{\partial V_{i,0}}{\partial S_0}$ is given by:*

$$q_S = \alpha_i V_{i,0} \left(m_i S_0^{m_i - 1} exp\left[1/2\ \sigma^2 m_i(m_i-1)T\right]\right) N\left[d_1\left(\widehat{S}_0, \widehat{K}, \widehat{\sigma}, \widehat{r}\right)\right], \quad (57)$$

where:

$$\begin{aligned} \widehat{S}_0 &= S_0^{m_i} exp\left[1/2\ \sigma^2 m_i(m_i-1)T\right], \widehat{K} = p_i/\alpha_i, \\ \widehat{\sigma} &= m_i\sigma, \ \text{and}\ \widehat{r} = r. \end{aligned}$$

Proof. The power call $Max\left[\alpha_i V_{i,0} S_T^{m_i} - p_i V_{i,0}, 0\right]$ is equal to

$$\alpha_i V_{i,0} Max\left[S_T^{m_i} - p_i/\alpha_i, 0\right].$$

Recall that we suppose:

$$S_t = S_0 exp\left[(\mu - 1/2\sigma^2)t + \sigma W_t\right].$$

Thus, we have:

$$S_T^{m_i} = S_0^{m_i} exp\left[m_i(\mu - 1/2\sigma^2)T + m_i\sigma W_T\right],$$

which is also equal to

$$S_0^{m_i} exp\left[1/2\ \sigma^2 m_i(m_i-1)T\right].exp\left[(m_i\mu - 1/2 m_i^2\sigma^2)T + m_i\sigma W_T\right].$$

Therefore, using the Black and Scholes formula for a standard call option with underlying $\widehat{S}_0$, strike $\widehat{K}$, volatility $\widehat{\sigma}$, interest rate $\widehat{r}$, the initial value of the power call is given by:

$$N\left[d_1\left(\widehat{S}_0, \widehat{K}, \widehat{\sigma}, \widehat{r}\right)\right] - N\left[d_2\left(\widehat{S}_0, \widehat{K}, \widehat{\sigma}, \widehat{r}\right)\right],$$

with:

$$\begin{aligned} d_1\left(\widehat{S}_0, \widehat{K}, \widehat{\sigma}, \widehat{r}\right) &= \frac{\ln\left(\frac{\widehat{S}_0}{\widehat{K}}\right) + \left(\widehat{r} + \frac{1}{2}\widehat{\sigma}^2\right)T}{\widehat{\sigma}\sqrt{T}}, \\ d_2\left(\widehat{S}_0, \widehat{K}, \widehat{\sigma}, \widehat{r}\right) &= d_1\left(\widehat{S}_0, \widehat{K}, \widehat{\sigma}, \widehat{r}\right) - \widehat{\sigma}\sqrt{T}, \end{aligned}$$

and:

$$\begin{aligned} \widehat{S}_0 &= S_0^{m_i} exp\left[1/2\ \sigma^2 m_i(m_i-1)T\right], \\ \widehat{K} &= p_i/\alpha_i, \\ \widehat{\sigma} &= m_i\sigma, \\ \widehat{r} &= r. \end{aligned}$$

Consequently, we have:

$$\frac{\partial V_{i,0}}{\partial S_0} = \alpha_i V_{i,0}\left(\frac{\partial \widehat{S}_0}{\partial S_0}.\frac{\partial V_{i,0}}{\partial \widehat{S}_0}\right).$$

Thus we have:

$$\frac{\partial V_{i,0}}{\partial S_0} = \alpha_i V_{i,0}\left(m_i S_0^{m_i-1} exp\left[1/2\ \sigma^2 m_i(m_i-1)T\right]\right) N\left[d_1\left(\widehat{S}_0, \widehat{K}, \widehat{\sigma}, \widehat{r}\right)\right].$$

■

We illustrate now numerically this particular case for our numerical base case (49). For the numerical base case, we get the following investor and banker profiles: the banker and the investor have CRRA utilities with respective RRA γ_b and γ_i. In Appendix (A.5), we illustrate both the investor's and banker's portfolio profiles. We consider both three values for the drift μ ($\mu = 4\%; \mu = 7\%; \mu = 15\%$) and for the volatility σ ($\sigma = 10\%; \sigma = 20\%; \sigma = 30\%$). Five levels of the investor's relative risk aversion (RRA) are introduced: $\gamma_i = 0.5$ and 2 ("aggressive investors"); $\gamma_i = 5$ (moderate investor); $\gamma_i = 7$ and 10 (more conservative investors). For the banker, we assume the following relative risk aversions: $\gamma_b = 0.1$, 0.5, 2, 5 and 10.

In Appendix 5, figures about both investor's and banker's payoffs are provided to show the influence of expected risky asset return μ and volatility σ on the portfolio profiles. We note for example that the banker's payoff is most of the time positive except for weak RRA γ_i of the investor and/or high risky expected return μ. However, for high risky expected return μ, it is still positive for small RRA γ_i (for example, $\gamma_i = 10$ in Figure A.5.3). For small volatility

($\sigma = 10\%$) and small RRA γ_i (for example, $\gamma_i = 0.5$ in Figure A.5.4), the banker's payoff is flat. Note also that most of the time, it decreasing from a given value of the spot price S_0.

We now examine the numerical values of the CV $\frac{\tilde{V}_{b,0}}{V_{b,0}}$. The numerical values of the CV $\frac{\tilde{V}_{b,0}}{V_{b,0}}$ are displayed in Table (3), which provides the CV values for the risk-neutral hedge criterion. We still set $V_{0,b} = 1000$. We can see that for high Sharpe type ratio due to high return ($\frac{\mu-r}{\sigma^2} = 3$ for the case $\mu = 15\%$ and $\sigma = 20\%$), the (theoretical) CV is very high (except for the less risky position corresponding to $\gamma_i = 10$ and $\gamma_b = 10$. In that case, the CV is equal to 5.4%). As for the VaR constraint, this is due to the high risk of the constrained banker's payoff, due to the convexity of the investors' payoff and/or his additional guarantee constraint.[16] For high Sharpe type ratio due to small volatility ($\frac{\mu-r}{\sigma^2} = 4$ for the case $\mu = 7\%$ and $\sigma = 10\%$), the CV lies between 6% and 100%. For the most convenient banker's RRA $\gamma_b = 10$, the CV value is about 12%. These CV values are smaller than previous ones when $\gamma_b = 10$, since the volatility risk is small. For more standard Sharpe type ratio ($\frac{\mu-r}{\sigma^2} = 1$ for the case $\mu = 7\%$ and $\sigma = 20\%$), the (theoretical) CV is can be very high for weak RRA γ_b of the banker ($\gamma_b = 0.1; 0.5$) and/or for weak RRA γ_i of the investor. When the banker's RRA γ_b is relatively high (more convenient case), for instance $\gamma_b = 10$, his CV is about 4% or 5%, except for the riskier case ($\gamma_i = 0.5$). For relatively small Sharpe type ratio ($\frac{\mu-r}{\sigma^2} = \frac{4}{9}$, for the case $\mu = 7\%$ and $\sigma = 30\%$), the CV is still high (about 20%) since the volatility is also very significant ($\sigma = 30\%$). Finally, for small Sharpe type ratio ($\frac{\mu-r}{\sigma^2} = \frac{1}{4}$, for the case $\mu = 4\%$ and $\sigma = 20\%$), the CV value lies mainly between 3% and 4.5%. As for the Var constraint case, this is due for instance to the bad performance of the financial market for the investor.

Note also that if we compute other CV numerical values, we get in particular the following values for the small volatility case ($\mu = 7\%$ and $\sigma = 10\%$):

$$CV[\gamma i = 0.5, \gamma_b = 0.1] \simeq 0\% \text{ and } CV[\gamma i = 0.5, \gamma_b = 0.5] = 7.28\%$$
$$CV[\gamma i = 2, \gamma_b = 0.1] \simeq 0\% \text{ and } CV[\gamma i = 2, \gamma_b = 0.5] = 7.4\%$$
$$CV[\gamma i = 5, \gamma_b = 0.1] \simeq 0\% \text{ and } CV[\gamma i = 5, \gamma_b = 0.5] = 8.4\%$$
$$CV[\gamma i = 10, \gamma_b = 0.1] \simeq 0\% \text{ and } CV[\gamma i = 10, \gamma_b = 0.5] = 9\%$$

The very weak CV values for $\gamma_b = 0.1$ are due to the better fit of the risk-neutral hedge banker's portfolio to his true optimal portfolio for small volatility when he has little relative risk-aversion.

[16] As in previous VaR case, the static risk-neutral hedge is rather inefficient in that case (recall that $T = 5$ years). Therefore, the banker's risk is too high, which implies very high CV to theoretically compensate this risk level.

Table 3: CV CRRA (banker) Risk-neutral hedge

RRA	$\mu = 4\%$ and $\sigma = 20\%$			$\mu = 7\%$ and $\sigma = 20\%$			$\mu = 15\%$ and $\sigma = 20\%$		
γ_i/γ_b	2	5	10	2	5	10	2	5	10
0.5	4.37%	4.35%	4.6%	47%	36%	42%	77%	93%	96%
2	3.7%	3.6%	3.54%	8.44%	5.9%	5.2%	75%	93%	96%
5	3.5%	3.33%	3.32%	8%	5.5%	4.7%	46%	20%	13%
10	3.4%	3.23%	3.21%	7.78%	5.1%	4.3%	45%	11.6%	5.4%
RRA	$\mu = 7\%$ and $\sigma = 10\%$			$\mu = 7\%$ and $\sigma = 20\%$			$\mu = 7\%$ and $\sigma = 30\%$		
γ_i/γ_b	2	5	10	0.1	0.5		2	5	10
0.5	68%	86%	91%	91%	87%		2.55%	2.51%	2.4%
2	69%	87%	92.8%	57%	20.5%		20%	20%	19.8%
5	78%	99%	5.5%	69.2%	21.6%		20.1%	20%	19.9%
10	84%	6%	12%	72.4%	21.15%		19.9%	19.9%	19.8%

4.3 Compensating variations of both the investor and the banker with respect to the standard OBPI case

In this subsection, we consider an investor who cannot exactly get his optimal portfolio. As emphasized in de Palma and Prigent (2008, 2009), typically financial institutions provide a limited number of standardized portfolios which do not exactly match investor preferences. Mistreating the "demand side" can also yield to significant utility losses for the investor. In what follows, we illustrate this feature for investors who can only buy the standard OBPI strategy. This financial product with guarantee is the first one (see Leland and Rubinstein, 1976), and serves as fundamental example to construct other guarantee funds.

We determine the compensating variation with respect to the true optimal payoff. The levels of compensating variation provide a measure of the monetary loss, due to of this type of friction. We also examine the CV for the banker in this framework.

The OBPI method consists basically in purchasing an amount $q_i.K_i$ invested on the money market account, and q_i shares of European call options written on asset S with maturity T and exercise price K_i. The value $V_{i,t}^{OBPI}$ of this portfolio at any time t in the period $[0, T]$ is:

$$V_{i,t}^{OBPI} = q_i \left(K_i e^{-r(T-t)} + C(t, K_i) \right), \tag{58}$$

where $C(t, x)$ denotes the no-arbitrage value of a European call option with strike x, calculated under a given risk-neutral probability Q (if coefficient functions μ, a and b are constant, $C(t, x)$ is the usual Black-Scholes value of the European call). Note that, for all dates t before T, the portfolio value $V_{i,t}^{OBPI}$ is always above the deterministic level $q_i K_i e^{-r(T-t)}$. The investor is still willing to recover a percentage p of his initial investment $V_{i,0}$. Then, the portfolio manager has to choose the two appropriate parameters, q_i and K_i.

- First, since the insured amount is equal to $q_i.K_i$, it is required that K_i satisfies the relation:[17]

$$p_i V_{i,0} = p_i q_i (K_i.e^{-rT} + C(0, K_i)) = q_i K_i, \tag{59}$$

which implies that:

$$\frac{C(0, K_i)}{K_i} = \frac{1 - p_i e^{-rT}}{p_i}. \tag{60}$$

Therefore, the strike K_i is an increasing function $K_i(p_i)$ of percentage p_i.

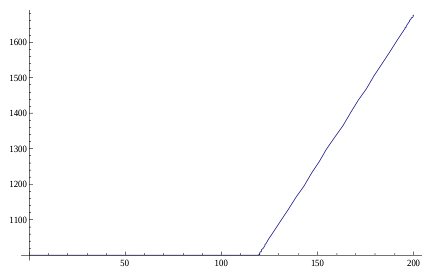
- Second, the number of shares q_i is given by:

$$q_i = \frac{V_{i,0}}{K_i e^{-rT} + C(0, K_i)}. \tag{61}$$

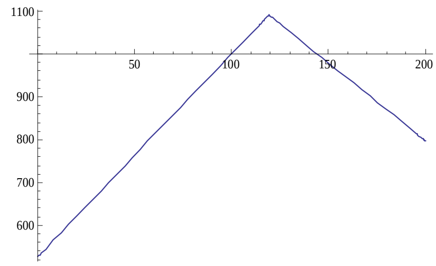
Thus, for any investment value $V_{i,0}$, number of shares q_i is a decreasing function of percentage p_i.

For our numerical base case (13), we get: $K_i = 119.3$ and $q_i = 8.38$.

Next figure illustrates both the investor's and banker's portfolio profiles when the OBPI portfolio is provided to the investor. For the banker, we consider the risk-neutral hedge portfolio with initial investment $V_{b,0} = 1000$.



OBPI investor's portfolio profile



OBPI banker's portfolio profile

Figure 8: OBPI Profiles

As it can be seen, the investor's guarantee ($V_{i,0} \geq V_{i,T}$) implies that the banker's constrained portfolio may suffer from significant losses for bearish market (for example if $S_T \leq 0.8\ S_0$,then the loss is higher than 10%) or when the risky asset price S_T grows highly ($S_T \geq 1.45\ S_0$).

[17]This relation can also be adjusted to take account of the smile effect.

We compute now the CV for both the investor and the banker, with same RRA values γ_i and γ_b as previously. Recall that the OBPI strategy is the optimal investor's portfolio if and only if $\gamma_i = 1/(\frac{\mu-r}{\sigma^2})$ (in that case, the power m_i is equal to 1). Apart this case, the investor does not receive his true optimal portfolio and bears utility losses. The banker also has not his true optimal portfolio. Additionally, he may suffer from portfolio value losses. Consequently, we must examine both their compensating variations. They are displayed in next table. We note that in this framework, the CV is generally much higher for the banker than the investor. However, if we consider the standard case (risky asset return $\mu = 7\%$, which is the usual value on long term horizon and volatility $\sigma = 20\%$), then the investor's and banker's compensating variations are almost equal. Roughly speaking, it means that these two CV compensate each other, which implies that only small CV values can be required by the banker (if any).

Table 4: CV for the CRRA case (OBPI)

RRA	$\sigma = 20\%$			$\mu = 7\%$		
γ_i/σ	$\mu = 4\%$	$\mu = 7\%$	$\mu = 15\%$	$\sigma = 10\%$	$\sigma = 20\%$	$\sigma = 30\%$
0.5	5.36%	15.2%	$\times$2.3	$\times$2.1	15.2%	19.4%
2	7.52%	6.5%	21.6%	20%	6.5%	19.7%
5	11.7%	9.2%	9%	10.3%	9.2%	24.5%
10	14.1%	12.7%	11%	10.1%	12.7%	28.6%
RRA	$\sigma = 20\%$			$\mu = 7\%$		
γ_b	$\mu = 4\%$	$\mu = 7\%$	$\mu = 15\%$	$\sigma = 10\%$	$\sigma = 20\%$	$\sigma = 30\%$
0.5	$\times$3.2	19.5%	$\times$6.2	$\times$4.2	19.5%	$\times$3.4
2	$\times$2.67	7.4%	$\times$5.9	$\times$3.9	7.4%	$\times$2.4
5	$\times$2.82	7.1%	$\times$6.3	$\times$5.6	7.1%	$\times$2.7
10	$\times$4.1	11.4%	$\times$6.5	5.2%	11.4%	$\times$3.1

5 Conclusion

Using the expected utility theory framework, we determine the optimal payoffs and their prices at equilibrium under insurance constraints on the horizon wealth for a large class of models. The results prove that derivative assets have to be introduced in the portfolio to maximize the expected utility of investors. The optimal solution clearly depends on the risk aversion of the investor and on the specification of the guarantee at maturity. Under the standard assumptions that the insurance constraints and the payoff are modelled by continuous functions of the risky asset, the solution is equal to the maximum between this function and the solution of the unconstrained problem but with a different initial wealth. These optimal portfolios can be determined for quite general utility functions, stock prices and insurance constraints. In the no guarantee case, the concavity/convexity of the portfolio profile is determined from the degree of risk aversion and from the financial market performance, for example a Sharpe type ratio. This kind of result still holds according to the insurance constraint at maturity. We show that, even with exogenous insurance constraints, the equilibrium risk-neutral density is equal to the product of a factor corresponding to the total risk tolerance, and a factor reflecting the personal beliefs, which is a generalization of Carr and Madan (2001). Note that the first factor is a positive decreasing function of the risky asset value which may imply a change in the mean and involve negative skewness. Then, we introduce the notion of compensating variation to provide a quantitative measure of the utility losses from not getting the true optimal portfolios. We provide the numerical illustration of these theoretical compensations for banker and investor having both CRRA utilities, through three main cases (investor having no direct access to the financial derivatives market; banker's compensating variation due to riskier financial position induced by the investor's guarantee; and finally the standard OBPI strategy as benchmark to both measure the investor's and banker's compensating variations). For standard financial parameter values and rational relative risk aversions (it means in particular that the banker must have a significant risk aversion), the level of these compensating variations lies between 3% and 10% for the banker's compensating variation, when he bears the insurance constraint of the investor. However, if we compute both the banker's and the investor's compensating variations with respect to the benchmark OBPI strategy, the two values are very close. In this latter case, this balance between the compensating variations yields to small compensating variations at the equilibrium.

References

[1] Arditti, F. and John, K. (1980). Spanning the state space with options. *Journal of Financial Quantitative Analysis*, 15, 1-9.

[2] Bazak, S., (1995). A general equilibrium model of portfolio insurance. *Review of Financial Studies,* 8, 1059-1090.

[3] Bazak S. and Shapiro A. (2001). Value-at-Risk based risk management: optimal policies and asset prices. *Review of Financial Studies*, 14, 371-405.

[4] Barrieu, P. and El Karoui, N. (2003). Optimal design of financial derivatives. *C.R. Acad. Sci. Paris*, Ser. I 336, 493-498.

[5] Benniga, S. and Blume, M. (1985). On the optimality of portfolio insurance. *Journal of Finance*, 40, 1341-1352.

[6] Benninga, S. and Mayshar, J., (2000). Heterogeneity and option pricing. *Review of Derivatives Research*, 40, 1341-1352.

[7] Bick, A., (1987). On the consistency of the Black-Scholes model with a general equilibrium framework. Journal of Financial and Quantitative Analysis, 22, 259-276.

[8] Bertrand, P., Lesne, J-P. and Prigent, J-L., (2001). Gestion de portefeuille avec garantie: l'allocation optimale en actifs dérivés. *Finance,* 22, 7-35.

[9] Bertrand, P. and Prigent, J-L., (2001). Portfolio insurance strategies: OBPI versus CPPI. *Finance*, 2005, 26, 5-32.

[10] Bertrand, P. and Prigent, J-L., (2003). Portfolio insurance strategies: a comparison of standard methods when the volatility of the stock is stochastic. *International Journal of Business,* 8, 461-472.

[11] Bertrand P. and Prigent, J-L., (2005). Portfolio insurance strategies: OBPI versus CPPI. *Finance*, 26, 5-32.

[12] Bertrand P. and Prigent, J.-L. (2010). A note on risk aversion, prudence and portfolio insurance, *Geneva Risk and Insurance Review*, 35, 81-92

[13] Bertrand, P. and Prigent, J-L., (2011). Omega performance measure and portfolio insurance. *Journal of Banking and Finance*, 35, 1811-1823.

[14] Black, F. and Jones, R. (1987). Simplifying portfolio insurance. *Journal of Portfolio Management,* 48-51.

[15] Black, F., and Rouhani, R. (1989). Constant proportion portfolio insurance and the synthetic put option : a comparison, in Institutional Investor focus on Investment Management, edited by Frank J. Fabozzi. Cambridge, Mass. : Ballinger, pp 695-708.

[16] Black, F. and Perold, A.R. (1992). Theory of constant proportion portfolio insurance. *Journal of Economics, Dynamics and Control*, 16, 403-426.

[17] Bookstaber, R. and Langsam, J.A. (2000). Portfolio insurance trading rules. *Journal of Futures Markets,* 8, 15-31.

[18] Brennan, M.J. and Schwartz, R. (1989). Portfolio insurance and financial market equilibrium. *Journal of Business*, 62, 455-472.

[19] Brennan, M.J. and Solanki, R., (1981). Optimal portfolio insurance. *Journal of Financial and Quantitative Analysis,* 16, 3, 279-300.

[20] Brady N. et al. (1988). Report of the presidential task force on market mechanisms. U.S. Government Printing Office, Washington D.C.

[21] Breeden, D. and Litzenberger, R., (1978). Prices of state contingent claims implicit in options prices. *Journal of Business,* 51, 621-651.

[22] Carr, P. and Madan, D., (2001). Optimal positioning in derivative securities. *Quantitative Finance*, 1, 19-37.

[23] Cass, D. and Stiglitz, J. (1970). The structure of investor preferences and asset returns and separability in portfolio allocation. *Journal of Economic Theory,* 2, 122-160.

[24] Cox, J. and Huang, C.-F., (1989). Optimal consumption and portfolio policies when assets prices follow a diffusion process. *Journal of Economic Theory,* 49, 33-83.

[25] Cvitanic, J. and Karatzas, I., (1996). Contingent claim valuation and hedging with constrained portfolio. IMA volume in Math, Davis M., Duffie D., Fleming, W. and Shreve, S. eds , 65, 13-33.

[26] Das S. (2000).Structured Products and Hybrid Securities. 2nd ed., New York, NY: John Wiley and Sons.

[27] De Palma, A., Picard N., and Prigent, J-L., (2009). Prise en compte de l'attitude face au risque dans le cadre de la directive MiFID. Ecole Polytechnique, WP 2009-35.

[28] De Palma, A. and Prigent, J-L., (2008).Utilitarianism and fairness in portfolio positioning, *Journal of Banking and Finance*, 32, 1648-1660.

[29] De Palma, A. and Prigent, J-L., (2009). Standardized versus customized portfolio: a compensating variation approach. *Annals of Operations Research*, 165, 161-185.

[30] Driessen, J. and P. Maenhout, (2007). An empirical portfolio perspective on option pricing anomalies. *Review of Finance*, 11, 561-603.

[31] Duffie, D. and Jackson, M. (1989). Optimal innovation of futures contracts. *Review of Financial Studies*, 2, 275-296.

[32] Eeckhoudt L. and Gollier, C., (2005). The impact of prudence on optimal prevention. *Economic Theory*, 26, 989-994.

[33] Ekeland, I. and Turnbull, T., (1983). *Infinite-Dimensional Optimization and Convexity.* Chicago Lectures in Mathematics, the University of Chicago Press.

[34] El Karoui, N., Jeanblanc, M. and Lacoste, V. (2005). Optimal portfolio management with American capital guarantee. *Journal of Economics, Dynamics and Control*, 29, 449-468.

[35] Franke, G., Stapleton, R. and Subrahmanyam, M., (1998). Who buys and who sells options: the role of options in an economy with background risk. *Journal of Economic Theory*, 82, 89-109.

[36] Friesen, P., (1979). The Arrow Debreu model extended to financial markets. *Econometrica*, 47, 689-707.

[37] Gollier, (2001). *The Economics of Risk and Time.* MIT Press.

[38] Grossman, S. and Vila, J.L., (1989). Portfolio insurance in complete markets: a note. *Journal of Business,* 60, 275-298.

[39] Grossman, S. and Zhou, J. (1996). Equilibrium analysis of portfolio insurance. *Journal of Finance,* 51, 1379-1403.

[40] Hakansson, N. (1978). Welfare aspects of options and supershares. *Journal of Finance*, 33, 759-776.

[41] Hens, T and M.O. Riger, (2008): The dark side of the moon: structured products from the customer's perspective. Working paper 459, ISB, Université de Zurich.

[42] Jensen, B.A. and C. Sorensen, (2001). Paying for minimum interest rate guarantees: who should compensate who ? *European Financial Management*, 7, 183-211.

[43] Johnston, L. and McConnell, J., (1989). Requiem for amarket: an analysis of the rise and fall of a financial futures contract. *Review of Financial Studies*, 2, 1-24.

[44] Karatzas, I., Lehoczky, J., Sethi, S.P. and Shreve, S.E., (1986). Explicit solution of a general consumption/investment problem. *Mathematics of Operations Research,* 11, 261-294.

[45] Karatzas, I. and Shreve, S.E., (1998). *Methods of Mathematical Finance.* Springer-Verlag, Berlin.

[46] Kreps, D., (1982). *Multiperiod Securities and the Efficient Allocation of Risk: A Comment on the Black-Scholes Option Pricing Model* (Chicago, IL: University of Chicago Press), Chapter 6, 203-32.

[47] Leland, H., (1980). Who should buy portfolio insurance? *Journal of Finance,* 35, 581-594.

[48] Leland, H.E. and Rubinstein, M. (1976). The evolution of portfolio insurance, in: D.L. Luskin, ed., Portfolio insurance: a guide to dynamic hedging, Wiley.

[49] Merton, R., (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3, 373-413.

[50] Perold, A. (1986). Constant portfolio insurance. Harvard Business School. Unpublished manuscript.

[51] Perold, A. and Sharpe, W. (1988). Dynamic strategies for asset allocation. *Financial Analyst Journal*, January-February, 16-27.

[52] Prigent, J-L., (2001). Assurance du portefeuille : analyse et extension de la méthode du coussin. *Bankers, Markets and Investors,* 51, 33-39.

[53] Prigent, J-L., (2007). *Portfolio Optimization and Performance Analysis.* Chapman & Hall: Boca Raton, Florida.

[54] Ross, S., (1976). Options and efficiency. *Quantitative Journal of Economy*, 90, 75-89.

[55] Stoimenov, P.A. and Wilkens, S. (2005). Are structured products "faily" priced? An analysis of the German market for equity-linked instruments. *Journal of Banking and Finance*, 29, 2971-2993.

[56] Tversky, A., and Kahneman, D., (1992). Advances in prospect theory: cumulative representation of uncertainty, *Journal of Risk and Uncertainty*, 5, 297-323.

Appendix

Appendix 1. (Optimal portfolio profile with guarantee constraint)

To solve this optimization problem, introduce the sets

$$K_{i,1} = \{h_i \in \mathbb{L}^2(\mathbb{R}^+, \mathbb{P}_i(ds)) | V_{i,0} = e^{-rT}\mathbb{E}_{\mathbb{P}_i}[h_i(S_T)M_{i,T}]\}$$

and

$$K_{i,2} = \{h_i \in \mathbb{L}^2(\mathbb{R}^+, \mathbb{P}_i(ds)) | h_i \geq h_{i,0}\}.$$

The set $K_i = K_{i,1} \cap K_{i,2}$ is a convex set of $\mathbb{L}^2(\mathbb{R}^+, \mathbb{P}_i(ds))$. Consider the following indicator function of K_i, denoted by δ_{K_i} and defined by:

$$\delta_{K_i}(h_i) = \begin{cases} 0 & if \quad h_i \in K_i \\ +\infty & if \quad h_i \notin K_i \end{cases}$$

Since K_i is closed and convex, δ_{K_i} is lower semi-continuous and convex.

Recall the notion of subdifferentiability: (see for example Ekeland and Turnbull (1983) for definition and properties of subdifferentials)

Let V denote a Banach space and $< ., . >$ the duality symbol.

Definitions:

1) For any function H defined on V with values in $\mathbb{R} \cup \{+\infty\}$, a continuous affine functional $l : V \to \mathbb{R}$ everywhere less than H (i.e. $\forall v \in V,\ l(v) \leq> H(v)$) is *exact* at v^* if $l(v^*) = H(v^*)$.

2) A function $H : V \to \mathbb{R} \cup \{+\infty\}$ is *subdifferentiable* at v^* if there exists a continuous affine functional $l(.) =< ., v_l > -a$, everywhere less than H, which is exact at v^*. The slope v_l of such an l is a *subgradient* of H at v^*. The set of all subgradients of H at v^* is the *subdifferential* of H at v^* and is denoted by $\partial H(v^*)$.

Recall the following characterization:

$$v_c \in \partial F(v^*) \text{ iff } F(v^*) < +\infty \text{ and } \forall v \in V, < v - v^*, v_c > +F(v^*) \leq F(v). \quad (62)$$

Denote by $\partial\delta_K$ the subdifferential of δ_K. The optimization problem is equivalent to:

$$Max_h \left(\mathbb{E}[U(h(X_T)] - \delta_K(h)\right) \quad (63)$$

The optimality conditions leads to the following result:

There exists a scalar λ_c and a function h_c defined on $\mathbb{L}^2(\mathbb{R}^+, \mathbb{P}(ds))$ such that:

$$h^* = J(\lambda_c g + h_c), \quad (64)$$

where λ_c is solution of the following problem: Find y such that

$$V_0 = \int_0^\infty J[yg(s) + h_c(s)]g(s)f(s)ds.$$

and $h_c \in \partial\delta_{K_2}(h^*)$.

Proof. Denote by $\mathcal{L}$ the functional defined by

$$\mathcal{L}(h) = (-\mathbb{E}[U(h(S_T))] + \delta_K(h)) .$$

The optimization problem

$$Max_h \left(\mathbb{E}[U(h(S_T)] - \delta_K(h)\right) \tag{65}$$

is equivalent to the following:

$$-Min_h \left(\mathcal{L}(h)\right), \tag{66}$$

where $\mathcal{L}$ is subdifferentiable and h^* is solution of the optimal problem iff $0 \in \partial\mathcal{L}(h^*)$.
Φ_U is continuous and, due to the kind of constraints, δ_K is continuous. Thus :

$$\partial\mathcal{L} = \partial(-\Phi_U + \delta_K) = \partial(-\Phi_U) + \partial(\delta_K).$$

Moreover, since Φ_U is differentiable then $\partial(\Phi_U) = \{\Phi'_U\}$. Additionally, we have:

$$\partial(\delta_K) = \partial(\delta_{K_1}) + \partial(\delta_{K_2}).$$

From the characterization (62), $h_1 \in \partial\delta_{K_1}(h^*)$ if and only if

$$\forall h, \int_0^\infty (h - h^*)(s) h_1(s) \mathbb{P}(ds) + \delta_{K_1}(h^*) \leq \delta_{K_1}(h).$$

In particular, $\forall h \in K_1, \int_0^\infty (h - h^*)(s) h_1(s) \mathbb{P}(ds) \leq 0$, from which we deduce that h_1 is orthogonal to any orthogonal function to the subspace generated by g. Thus, there exists a scalar λ_c such that $h_1 = \lambda_c g$.
To conclude, $0 \in \partial L(h^*)$ iff there exists $h_c \in \partial\delta_{K_2}(h^*)$ such that

$$0 = -U'(h^*) + \lambda_c g + h_c.$$

■

To explain more precisely the condition $h_c \in \partial\delta_{K_2}(h^*)$, assume that the insurance constraint h_0 and h_c are continuous.[18] Consequently, the optimal payoff h^* is continuous. Then, we deduce the following corollary:

Under the above assumption, the function h_c satisfies the following property:
1) If on an product I of intervals of values of S_T, $h^*(S_T) > h_0(S_T)$ then h_c is equal to 0 on I.
2) If on an product I of intervals of values of $S_T, h^*(S_T) = h_0(S_T)$ then h_c is negative on I.

Proof. Recall that from the characterization (62), we deduce:

$$\forall h \in \mathbb{L}^2(\mathbb{R}^+, \mathbb{P}(ds)), \int_0^\infty (h - h^*)(s)(h_c)(s) \mathbb{P}(ds) + \delta_{K_2}(h^*) \leq \delta_{K_2}(h).$$

[18] Such properties are always verified in practice.

In particular :

$$\forall h \in \mathbb{L}^2(\mathbb{R}^+, \mathbb{P}(ds)) \text{ such that } h \geq h_0, \int_0^\infty (h - h^*)(s) h_c(s) \mathbb{P}(ds) \leq 0.$$

From the previous condition, we deduce:
If $\forall s \in I, h^*(s) > h_0(s)$ then, on each compact subinterval I_{co} of I, we can consider m_{co} equal to the minimum of $h^* - h_0$ on $I_{co}.m_{co}$ is non negative. Now, consider the function h equal to $h^* - (m_{co} 1_{I_{co}})$. By construction, $h > h_0$ everywhere. Therefore, the relation $\int (h - h^*)(s) h_c(s) \mathbb{P}(ds) \leq 0$ is true.
This implies :

$$m_{co} \int_{I_{co}} h_c(s) \mathbb{P}(ds) \geq 0.$$

On the other hand, by letting h equal to $h^* + a$ where a is a non negative constant, $a \int_{I_{co}} h_c(s) \mathbb{P}(ds) \leq 0$. Consequently, since for all I_{co}, $\int_{I_{co}} h_c(s) \mathbb{P}(ds) = 0$, h_c is equal to 0 on the product of intervals I.

To prove (2), consider for all I_{co}, a function h equal to $h^* + a 1_{I_{co}}$. Then, the relation $\int (h - h^*)(s) h_c(s) \mathbb{P}(ds) \leq 0$ becomes $a \int_{I_{co}} h_c(s) \mathbb{P}(ds) \leq 0$, from which the negativity of h_c on I is deduced. ∎

Under the previous assumptions on the utility U, there is one and only one continuous optimal payoff, associated to the unique solution λ_c of the budget equation.

Proof. From the assumptions on the marginal utility U', we deduce that its inverse J is a continuous and decreasing function with:

$$lim_{o^+} J = +\infty \text{ and } lim_{+\infty} J = 0$$

Thus, for all s, the function $\lambda_c \longrightarrow h^*(\lambda_c, s) = Max(h_0(s), \tilde{h}(\lambda_c, s))$ is continuous and decreasing. Therefore, the function $\lambda_c \longrightarrow \mathbb{E}_{\mathbb{Q}}[h^*(\lambda_c, S_T)]$ is continuous and decreasing from $+\infty$ to $\mathbb{E}_{\mathbb{Q}}[h_0(\lambda_c, S_T)]$ which is lower than the initial investment V_0. From the intermediate values theorem and by monotonicity, the result is deduced. ∎

Appendix 2. (European Guarantee: alternative proof)

The guarantee constraint consists in letting the portfolio value $V_{i,T}$ at maturity above a floor $F_{i,T}$. This floor may be deterministic, corresponding for example to a predetermined percentage p_i of the initial investment $V_{i,0}$ or may be stochastic if, for instance, the investor wants to benefit from potential market rises. For example, this floor may be equal to

$$F_{i,T} = a_i S_T + b_i,$$

where a_i is a given percentage of the benchmark S (a stock index for instance) and b_i is a fixed guaranteed amount which corresponds usually to a fixed percentage of the initial investment. In all cases, it is assumed that there exists a self-financing portfolio that duplicates the floor $F_{i,T}$.

Then, for a given initial investment V_0^*, the investor wants to find the portfolio θ solution of the following optimization problem:

$$\text{Max}_\theta \ \mathbb{E}_{\mathbb{P}_i}[U_i(V_{i,T})] \text{ under } V_{i,T} \geq F_{i,T}. \tag{67}$$

Due to market completeness, this problem is equivalent to (see Cox and Huang, 1989):

$$\begin{aligned} & Max_{V_{i,T}} \ \mathbb{E}_{\mathbb{P}_i}[U_i(V_{i,T})] \qquad (68) \\ \textit{under } V_{i,T} \quad \geq \quad & F_{i,T} \textit{ and } V_{i,0}^* = e^{-rT}\mathbb{E}_{\mathbb{P}_i}[V_{i,T}M_T] \geq e^{-rT}\mathbb{E}_{\mathbb{P}_i}[F_{i,T}M_{i,T}]. \end{aligned}$$

Then, we deduce:

The optimal solution $V_{i,T}^{**}$ of problem (67) is given by the maximum of the floor $F_{i,T}$ and the solution $V_{i,T}^*$ of the non constrained problem for an initial investment $V_{i,0}^*$ such that $V_{i,0}^* = e^{-rT}E_{\mathbb{P}_i}[Max(V_{i,T}^*, F_{i,T})M_{i,T}]$. Equivalently, this solution can be viewed as a combination of the portfolio value $V_{i,T}^*$ and a Put written on it with "strike" equal to the floor or a combination of the floor and a Call written on the portfolio value $V_{i,T}^*$.

$$V_{i,T}^{**} = V_{i,T}^* + (F_{i,T} - V_{i,T}^*)^+ = F_{i,T} + (V_{i,T}^* - F_{i,T})^+.$$

Proof. The proof is similar to KJL (2005). Consider the solution $V_{i,T}^*$ of the free problem (without guarantee constraint). Using Cox and Huang (1989), this solution is given by

$$V_{i,T}^* = J_i(\alpha_i M_{i,T}),$$

where the Lagrangian parameter α_i is such that $V_{i,0}^* = e^{-rT}\mathbb{E}_{\mathbb{P}_i}[V_{i,T}^* M_{i,T}]$.

Furthermore, for any portfolio $V_{i,T}$ with initial investment $V_{i,0}$ satisfying $V_{i,T} \geq F_{i,T}$, since the marginal utility U_i' is concave, we have :

$$U_i(V_{i,T}) - U_i(V_{i,T}^*) \leq U_i'(V_{i,T}^*)(V_{i,T} - V_{i,T}^*),$$

and, since $U_i\prime$ is decreasing, we deduce:

$$U_i'(V_{i,T}^*)(V_{i,T} - V_{i,T}^*) = \text{Min}(\alpha_i M_{i,T}, U_i'(F_{i,T}))(V_{i,T} - V_{i,T}^*).$$

Additionally,

$$\text{Min}(\alpha_i M_{i,T}, U_i'(F_{i,T}))(V_{i,T} - V_{i,T}^*) = \alpha_i M_{i,T}(V_{i,T} - V_{i,T}^*) - [\alpha_i M_{i,T} - U_i'(F_{i,T})]^+(V_{i,T} - F_{i,T}).$$

Finally, since $\mathbb{E}_{P_i}[V_{i,T}M_{i,T}] = V_{i,0}^* = \mathbb{E}_{\mathbb{P}_i}[V_{i,T}^* M_{i,T}]$, we get:

$$\mathbb{E}_{\mathbb{P}_i}[\text{Min}(\alpha_i M_{i,T}, U_i'(F_{i,T}))(V_{i,T} - V_{i,T}^*)] = -\mathbb{E}_{\mathbb{P}_i}[[\alpha_i M_{i,T} - U_i'(F_{i,T})]^+(V_{i,T} - F_{i,T})] \leq 0.$$

Therefore:

$$\mathbb{E}_{\mathbb{P}_i}[U_i(V_{i,T})] \leq \mathbb{E}_{\mathbb{P}_i}[U_i(V_{i,T}^*)].$$

∎

Appendix 3. (Optimal portfolio profiles at the equilibrium)

Proof of Corollary (20).

Recall that under homogeneous beliefs, the optimal portfolio profiles satisfy:

$$\frac{d}{ds}\left[h_i^{**}(s)\right] = \begin{array}{ll} h'_{i,g}(s) & \text{if } h_i^{**}(s) < h_{i,g}(s), \\ \frac{T_{i,o}(h_i(s))}{T(s)}, & \text{if } h_i^{**}(s) > h_{i,g}(s). \end{array} \tag{69}$$

Due to the linear risk tolerance, we get:

$$\frac{d}{ds}\left[h_1^{**}(s)\right] = h'_{1,g}(s), \text{ if } h_1^{**}(s) < h_{1,g}(s). \tag{70}$$

$$\frac{d}{ds}\left[h_2^{**}(s)\right] = h'_{2,g}(s), \text{ if } h_2^{**}(s) < h_{2,g}(s). \tag{71}$$

$$\frac{d}{ds}\left[h_1^{**}(s)\right] = \frac{\tau_1 + b_1 h_1^{**}(s)}{\tau + b_1 h_1^{**}(s) + b_2 h_2^{**}(s)}, \text{ if } h_1^{**}(s) > h_{1,g}(s). \tag{72}$$

$$\frac{d}{ds}\left[h_2^{**}(s)\right] = \frac{\tau_2 + b_2 h_2^{**}(s)}{\tau + b_1 h_1^{**}(s) + b_2 h_2^{**}(s)}, \text{ if } h_2^{**}(s) > h_{2,g}(s). \tag{73}$$

This system of ordinary differential equations can be explicitly solved if the cautiousness are opposite $b_1 = b = -b_2$ (see for example, Carr and Madan, 2001). Indeed, we have:

Case 1: if $h_1^{**}(s) > h_{1,g}(s)$ and $h_2^{**}(s) > h_{2,g}(s)$,

$$\frac{h_1'^{**}(s)}{h_2'^{**}(s)} = \frac{\tau_1 + b h_1^{**}(s)}{\tau_2 - b h_2^{**}(s)}.$$

Substituting $h_1^{**}(s) = s - h_2^{**}(s)$ leads to a quadratic equation for $h_2^{**}(s)$:

$$\frac{1}{2}\left[h_2^{**}(s)\right]^2 - \left[\frac{s}{2} + \frac{\tau}{2b}\right] h_2^{**}(s) + \frac{\tau_2 s + \eta}{2b} = 0,$$

with solution:

$$h_2^{**}(s) = \frac{s}{2} + \frac{\tau}{2b} - \sqrt{\left[\frac{s}{2} + \frac{\tau}{2b}\right]^2 - \frac{\tau_2 s + \eta}{2b}},$$

from which we deduce:

$$h_1^{*}(s) = \frac{s}{2} - \frac{\tau}{2b} + \sqrt{\left[\frac{s}{2} + \frac{\tau}{2b}\right]^2 - \frac{\tau_2 s + \eta}{2b}}.$$

We must set $c \leq \frac{\tau_1 \tau_2}{b}$. Then, let k be defined from equation:

$$\eta = \frac{\tau_1 \tau_2}{b} - c^2.$$

We finally deduce that the optimal payoffs are given by:

$$\begin{aligned} h_1^{**}(s) &= \frac{s}{2} - \frac{\tau}{2b} + \sqrt{\left(\frac{s}{2} + \frac{\tau_2 - \tau_1}{2b}\right) + c^2}, \\ h_2^{**}(s) &= \frac{s}{2} - \frac{\tau}{2b} - \sqrt{\left(\frac{s}{2} + \frac{\tau_2 - \tau_1}{2b}\right) + c^2}. \end{aligned} \tag{74}$$

Case 2: if $h_1^{**}(s) < h_{1,g}(s)$ and $h_2^{**}(s) < h_{2,g}(s)$,

$$\begin{aligned} h_1^{**}(s) &= h_{1,g}(s), \\ h_2^{**}(s) &= h_{2,g}(s). \end{aligned} \tag{75}$$

Case 3: if $h_1^{**}(s) > h_{1,g}(s)$ and $h_2^{**}(s) < h_{2,g}(s)$,

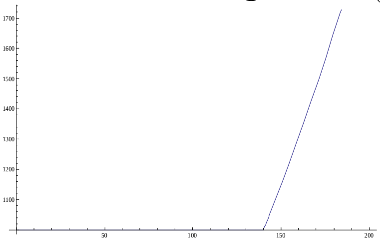
$$\begin{aligned} h_1^{**}(s) &= s - h_{2,g}(s), \\ h_2^{**}(s) &= h_{2,g}(s). \end{aligned} \tag{76}$$

Case 4: if $h_1^{**}(s) < h_{1,g}(s)$ and $h_2^{**}(s) > h_{2,g}(s)$,

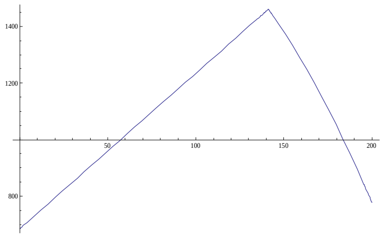
$$\begin{aligned} h_1^{**}(s) &= h_{1,g}(s), \\ h_2^{**}(s) &= s - h_{1,g}(s). \end{aligned} \tag{77}$$

Appendix 4. (Graphical illustrations of the investor's and bankers's payoffs for VaR constraints)

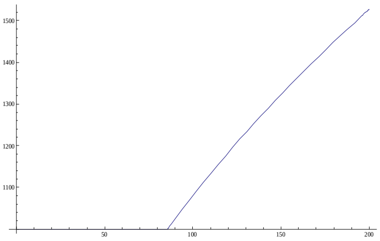
Fig. A.4.1. $(\mu = 7\%, \sigma = 20\%)$



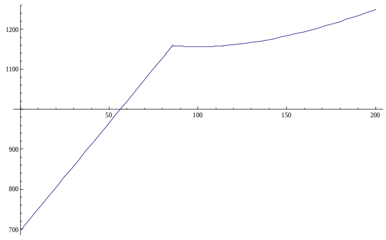
The investor's optimal profile $(\gamma_i = 0.5)$



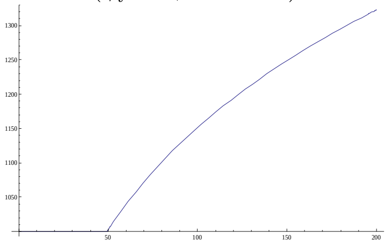
The banker's constrained profile $q_S = 5.5, q_B = 1450)$



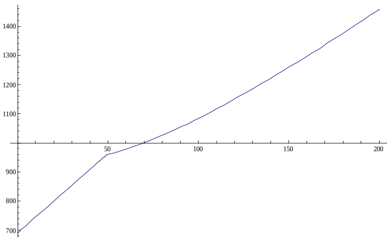
The investor's optimal profile $(\gamma_i = 2; m_i = 0.5)$



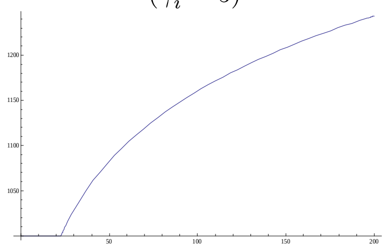
The banker's constrained profile $(q_S = 5.4, q_B = 1460)$



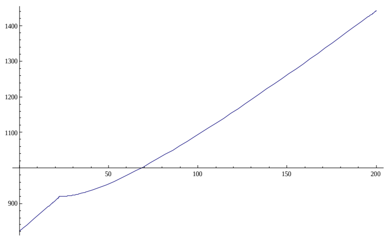
The investor's optimal profile $(\gamma_i = 5)$



The banker's constrained profile $(q_S = 5.45, q_B = 1455)$

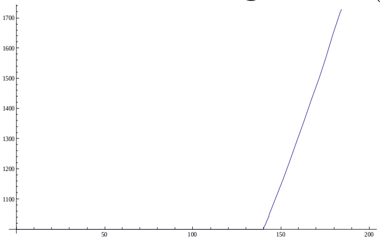


The investor's optimal profile $(\gamma_i = 10)$

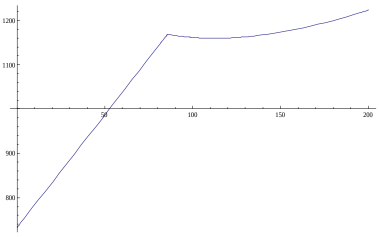


The banker's constrained profile $(q_S = 4.32, q_B = 1568)$

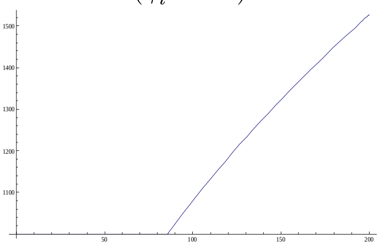
Fig. A.4.2. ($\mu = 4\%, \sigma = 20\%$)



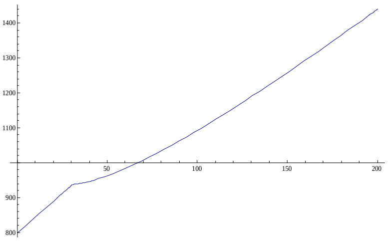
The investor's optimal profile
($\gamma_i = 0.5$)



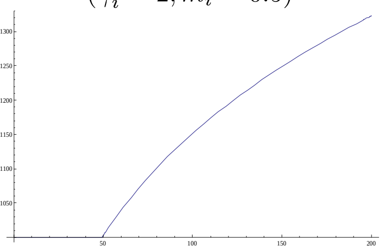
The banker's constrained profile
($q_S = 5.09, q_B = 1491$)



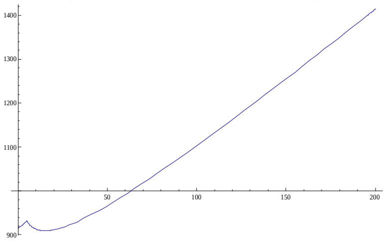
The investor's optimal profile
($\gamma_i = 2; m_i = 0.5$)



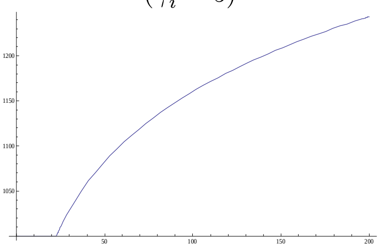
The banker's constrained profile
($q_S = 4.52, q_B = 1548$)



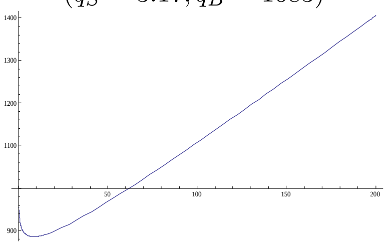
The investor's optimal profile
($\gamma_i = 5$)



The banker's constrained profile
($q_S = 3.17, q_B = 1683$)

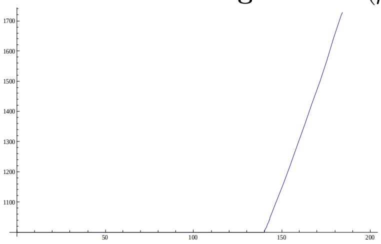


The investor's optimal profile
($\gamma_i = 10$)

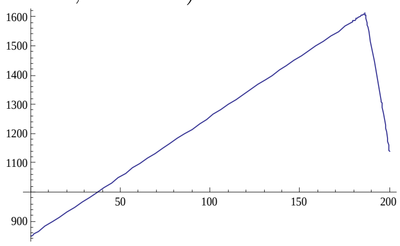


The banker's constrained profile
($q_S = 3.20, q_B = 16.80$)

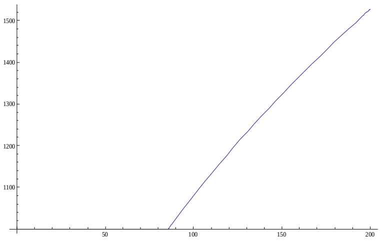
Fig. A.4.3. ($\mu = 15\%, \sigma = 20\%$)



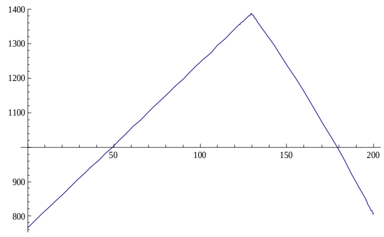
The investor's optimal profile
($\gamma_i = 0.5$)



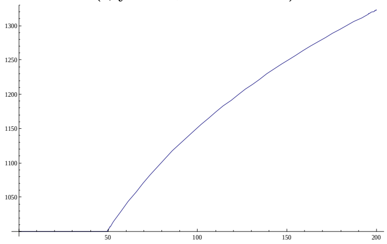
The banker's constrained profile
($q_S = 4.1, q_B = 1590$)



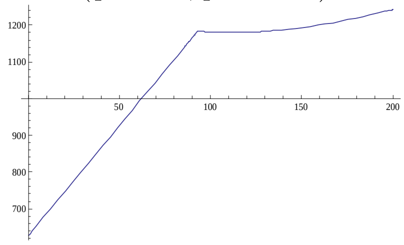
The investor's optimal profile
($\gamma_i = 2; m_i = 0.5$)



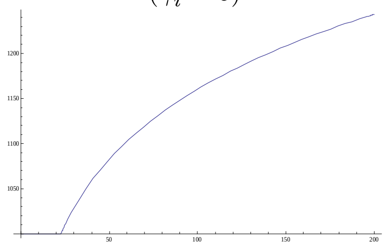
The banker's constrained profile
($q_S = 4.8, q_B = 1520$)



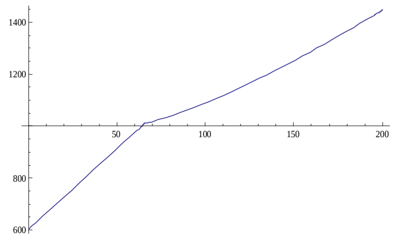
The investor's optimal profile
($\gamma_i = 5$)



The banker's constrained profile
($q_S = 6, q_B = 1400$)

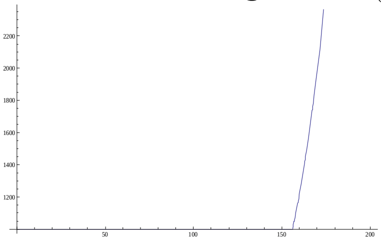


The investor's optimal profile
($\gamma_i = 10$)

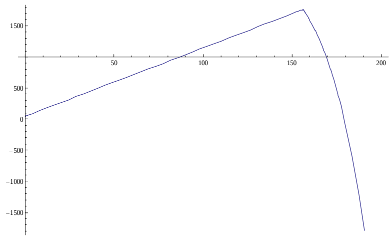


The banker's constrained profile
($q_S = 6.2, q_B = 1480$)

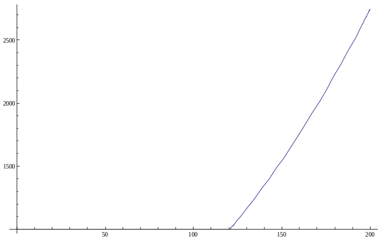
Fig. A.4.4. ($\mu = 7\%, \sigma = 10\%$)



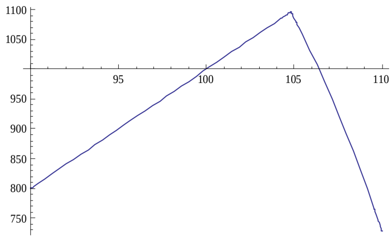
The investor's optimal profile
($\gamma_i = 0.5$)



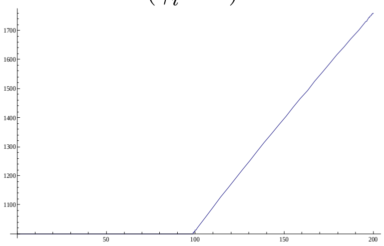
The banker's constrained profile
($q_S = 11, q_B = 900$)



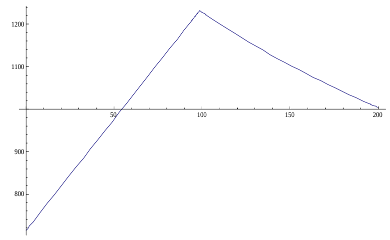
The investor's optimal profile
($\gamma_i = 2$)



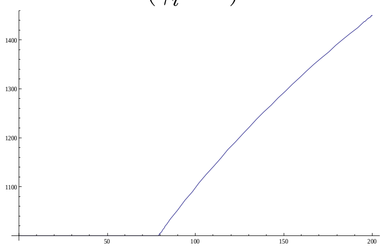
The banker's constrained profile
($q_S = 20, q_B = 0$)



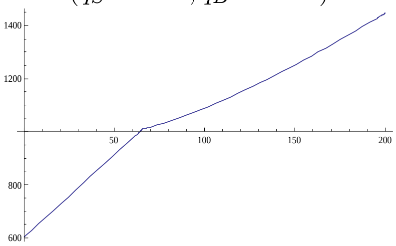
The investor's optimal profile
($\gamma_i = 5$)



The banker's constrained profile
($q_S = 5.25, q_B = 1475$)

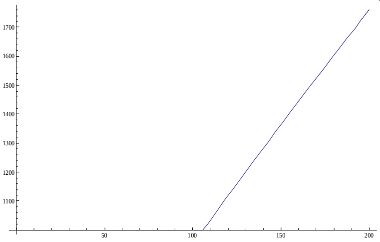


The investor's optimal profile
($\gamma_i = 10$)

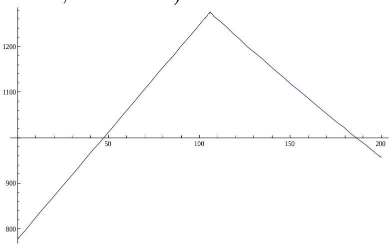


The banker's constrained profile
($q_S = 10.6, q_B = 940$)

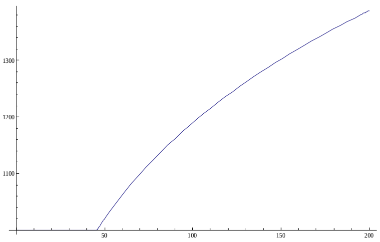
Fig. A.4.5. $(\mu = 7\%, \sigma = 30\%)$



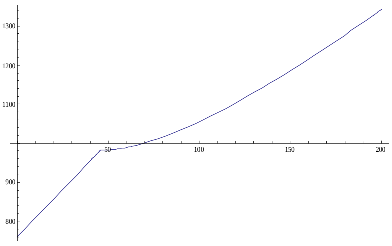
The investor's optimal profile
$(\gamma_i = 0.5)$



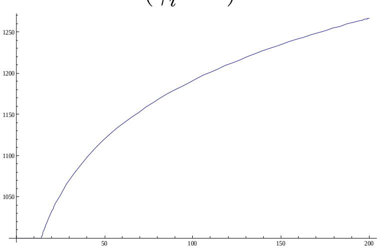
The banker's constrained profile
$(q_S = 4.7, q_B = 1530)$



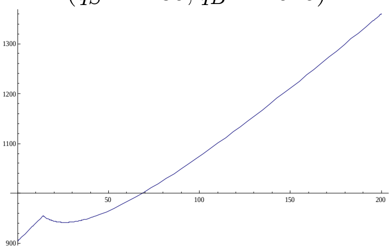
The investor's optimal profile
$(\gamma_i = 2)$



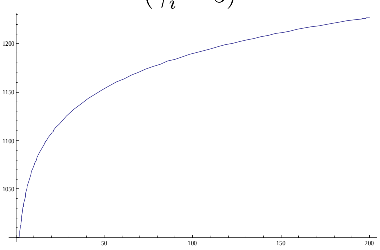
The banker's constrained profile
$(q_S = 4.85, q_B = 1515)$



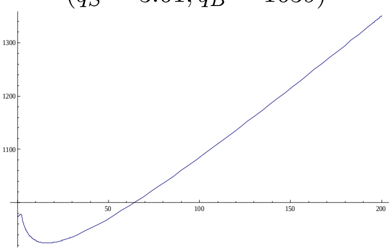
The investor's optimal profile
$(\gamma_i = 5)$



The banker's constrained profile
$(q_S = 3.61, q_B = 1639)$



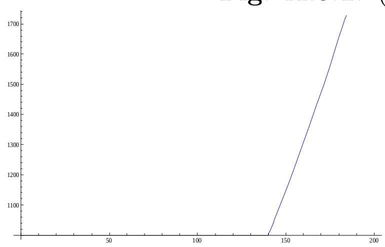
The investor's optimal profile
$(\gamma_i = 10)$



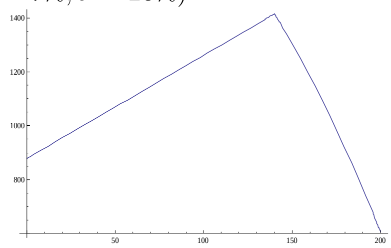
The banker's constrained profile
$(q_S = 3.02, q_B = 1698)$

Appendix 5. (Graphical illustrations of the investor's and bankers's payoffs for delta-neutrality constraints)

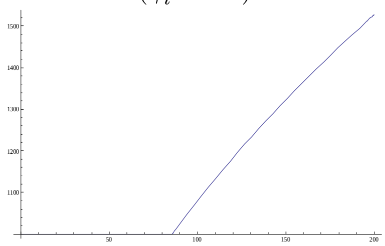
Fig. A.5.1. $(\mu = 7\%, \sigma = 20\%)$



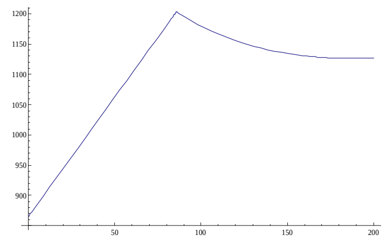
The investor's optimal profile ($\gamma_i = 0.5$)



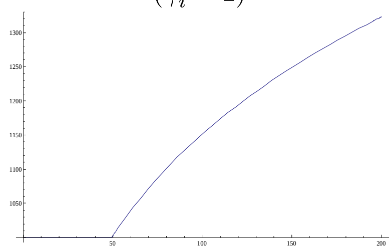
The banker's constrained profile ($q_S = 3.84, q_B = 1616$)



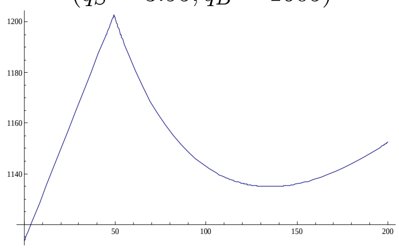
The investor's optimal profile ($\gamma_i = 2$)



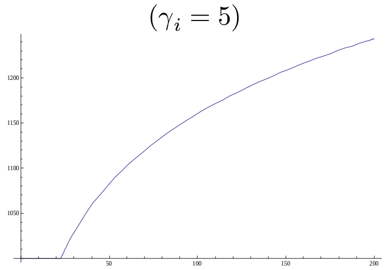
The banker's constrained profile ($q_S = 3.95, q_B = 1605$)



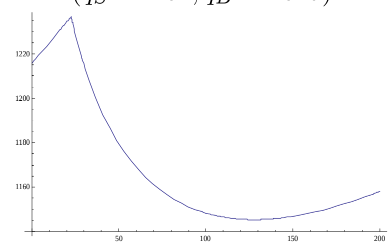
The investor's optimal profile ($\gamma_i = 5$)



The banker's constrained profile ($q_S = 1.81, q_B = 1819$)

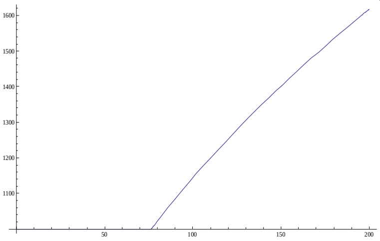


The investor's optimal profile ($\gamma_i = 10$)

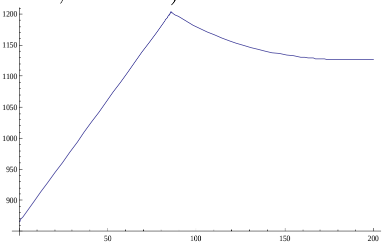


The banker's constrained profile ($q_S = 0.93, q_B = 1907$)

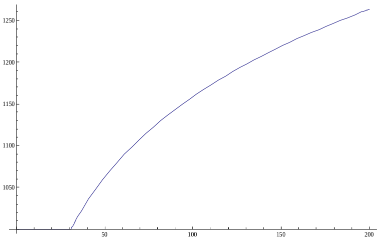
Fig. A.5.2. $(\mu = 4\%, \sigma = 20\%)$



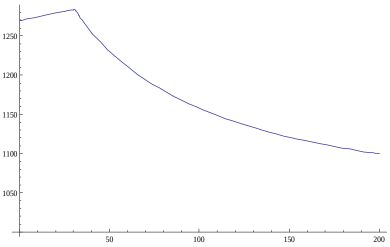
The investor's optimal profile
$(\gamma_i = 0.5)$



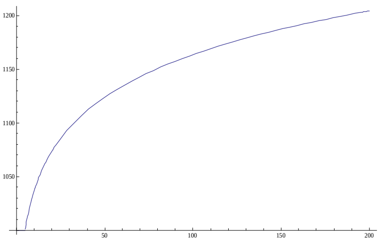
The banker's constrained profile
$(q_S = 3.95, q_B = 1605)$



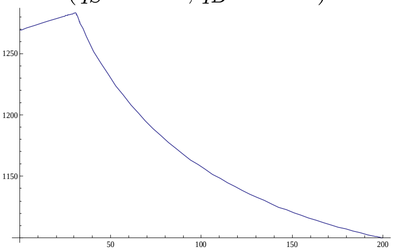
The investor's optimal profile
$(\gamma_i = 2)$



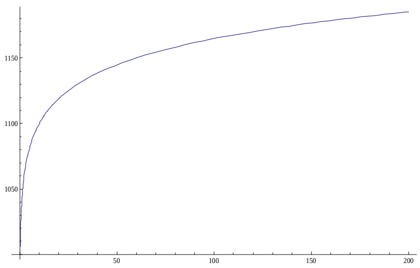
The banker's constrained profile
$(q_S = 1.15, q_B = 1885)$



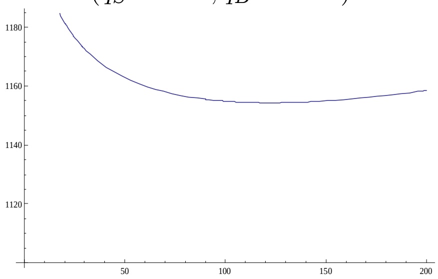
The investor's optimal profile
$(\gamma_i = 5)$



The banker's constrained profile
$(q_S = 0.47, q_B = 1953)$

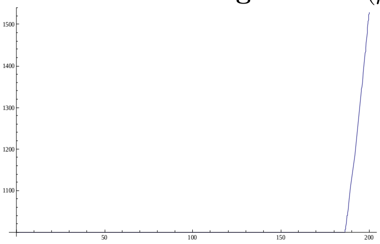


The investor's optimal profile
$(\gamma_i = 10)$

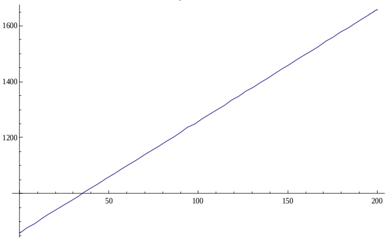


The banker's constrained profile
$(q_S = 0.24, q_B = 1976)$

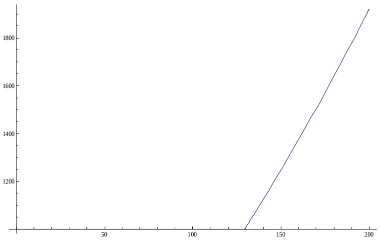
Fig. A.5.3. ($\mu = 15\%, \sigma = 20\%$)



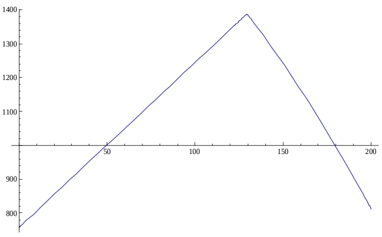
The investor's optimal profile
($\gamma_i = 0.5$)



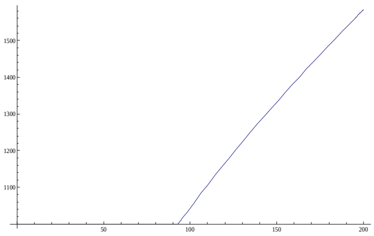
The banker's constrained profile
($q_S = 4, q_B = 1600$)



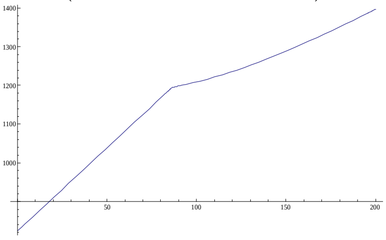
The investor's optimal profile
($\gamma_i = 2$)



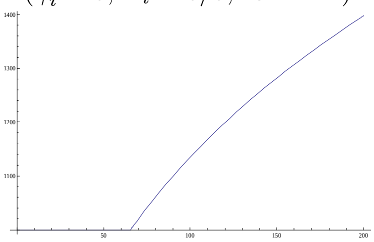
The banker's constrained profile
($q_S = 4.88, q_B = 1512$)



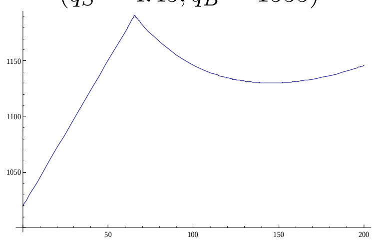
The investor's optimal profile
($\gamma_i = 5; m_i = 3/5$, "$OBPI$")



The banker's constrained profile
($q_S = 4.45, q_B = 1555$)

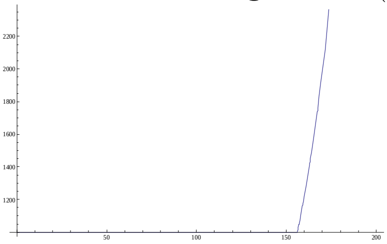


The investor's optimal profile
($\gamma_i = 10$)

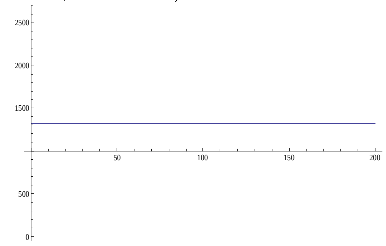


The banker's constrained profile
($q_S = 2.62, q_B = 1738$)

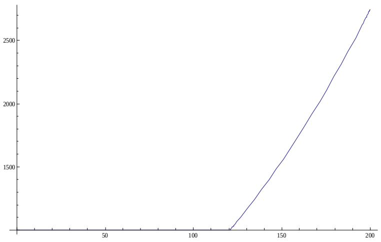
Fig. A.5.4. ($\mu = 7\%, \sigma = 10\%$)



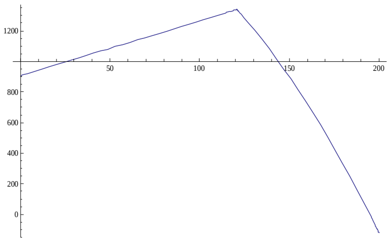
The investor's optimal profile
($\gamma_i = 0.5$)



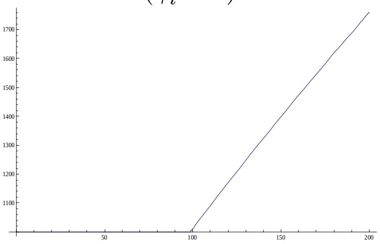
The banker's constrained profile
$q_S = 0, q_B = 2000$)



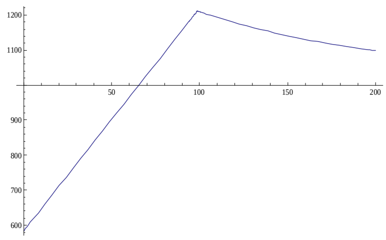
The investor's optimal profile
($\gamma_i = 2$)



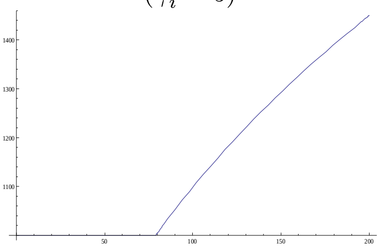
The banker's constrained profile
($q_S = 3.6, q_B = 1640$)



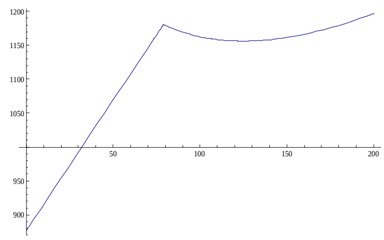
The investor's optimal profile
($\gamma_i = 5$)



The banker's constrained profile
($q_S = 6.38, q_B = 1362$)

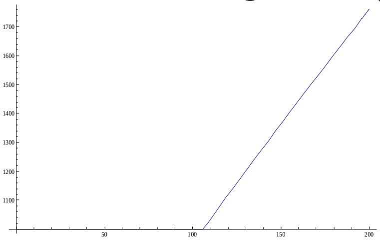


The investor's optimal profile
($\gamma_i = 10$)

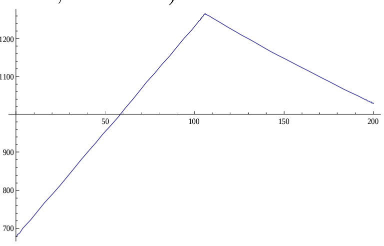


The banker's constrained profile
($q_S = 3.85, q_B = 1615$)

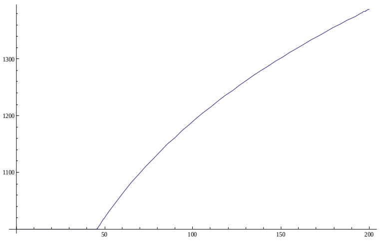
Fig. A.5.5. $(\mu = 7\%, \sigma = 30\%)$



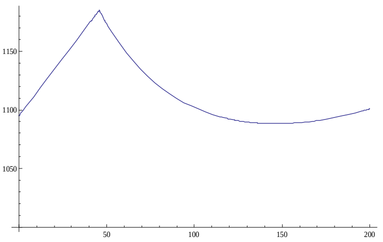
The investor's optimal profile
$(\gamma_i = 0.5)$



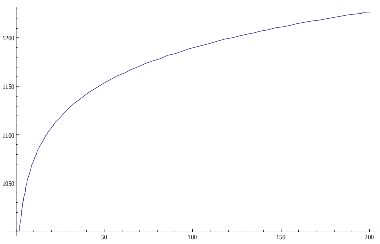
The banker's constrained profile
$q_S = 5.57, q_B = 1443)$



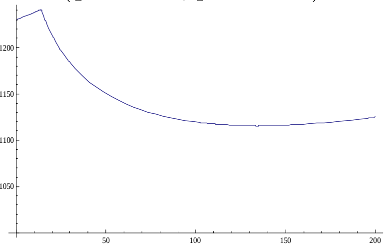
The investor's optimal profile
$(\gamma_i = 2)$



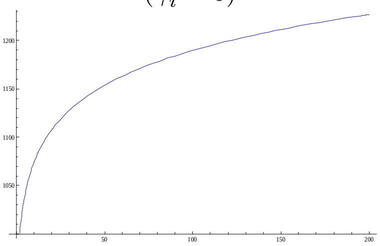
The banker's constrained profile
$(q_S = 1.97, q_B = 1803)$



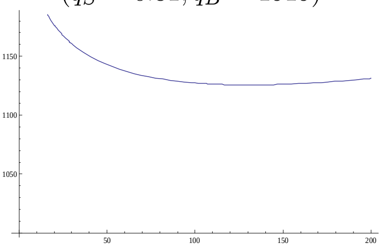
The investor's optimal profile
$(\gamma_i = 5)$



The banker's constrained profile
$(q_S = 0.81, q_B = 1919)$



The investor's optimal profile
$(\gamma_i = 10)$



The banker's constrained profile
$(q_S = 0.41, q_B = 1959)$