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Abstract

This paper deals with residential real estate portfolio optimization under taxation. In this framework, we examine an important decision making problem, namely the determination of the optimal time to sell a diversified real estate. The optimization problem corresponds to the maximization of the expectation of both the free cash flows and the terminal value of the portfolio. Taking account of the various taxes with several specific degression functions on the capital gains, we examine how the optimal time to sell behaves according to various financial parameter values and taxation levels. We show that the introduction of taxation highly modifies the structure of the optimal time to sell the real estate asset. For example, this latter one can jump whereas the expectation of the global wealth is a continuous function with respect to time. Additionally, contrary to the case with no taxation, the optimal time to sell depends on the volatility of the real estate asset, since the capital gains can be viewed as a real option involving different Call options. We provide numerical illustrations to emphasize such features and to examine the impact of various market and taxation parameters. We also compare the impact of different degression functions on the capital gains for four European countries, namely Germany, England, France and Spain.[1]

Key Words Real estate portfolio; Residential real estate; Optimal holding period; Taxation.

JEL Classification C61, G11, R21

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1. Introduction

The determination of the optimal holding period is a standard topic in finance. Many theoretical and empirical research studies deal with this problem (see Atkins and Dyl, 1997; In, Kim and Gençayc, 2011 and Lim and Kim, 2011). Holding period is defined as the expected period of time during which an investment is or should be kept. For a long time, this problem has been mainly studied for stocks. For instance, Demsetz (1968) and Tinic (1972) noted that transaction costs have a non negligible impact on holding periods. Amihud and Mendelson (1986) emphasize that assets having high bid-ask spreads (a proxy for high transaction costs) are kept at equilibrium by investors who expect to hold assets for a long time. Using a sample of NASDAQ and NYSE firms from 1981 to 1993, Atkin and Dyl (1997) examine the effects of firm size, bid-ask spread and volatility of returns on holding periods of stocks. They exhibit a positive correlation between the holding period, transaction costs and firm size, and a negative correlation between the holding period and price variability. Real estate assets exhibit these two features of high transaction costs and significant asset specific volatility, which is precisely why the optimal holding period represents a challenge both for academics and practitioners in the field.

The literature about the search for the optimal holding period in real estate portfolio management is rather recent. Usually, real estate investment is viewed as a rather passive process, since investor's strategy is very often reduced to a simple buy-and-hold strategy for real estate, this latter one ensuring relatively stable cash flows for many years. Such strategy avoids too large transaction costs but, since real estate cycles tend to be shorter, it is worthwhile to better analyze the concept of holding period and especially of ex-ante holding period.[2] This has been the subject of many empirical studies, although no consensus has emerged. For the USA case, Hendershott and Ling (1984) and Gau and Wang (1994) emphasize tax laws condition holding durations. For the UK case, the relationship between returns and holding periods appears to be complex. In a study based on interviews with investors, Rowley et al. (1998) show that investors and property developers have a specific holding period in mind from the start. They conclude that, for office space, a holding period decision is linked to depreciation and obsolescence. For retail property, the decision is more empirical, depending on active asset management and the prevailing situation in the commercial property market. In a more recent paper, Collett et al. (2003) highlight the fact that setting a holding period for the investor is important for any decision to invest in commercial real estate portfolios. Investment valuation requires a specified analysis period, and asset allocation depends on the variances and covariances of assets in turn influenced by a reference interval. Using the UK database of properties provided by IPD over an 18-year period, they observe that the median holding period is about seven years. Sales rates vary across the holding period (probably due to rent cycles and lease structures), and the holding period varies by property type: the larger and more expensive the properties, the longer the holding period. If the return is greater, the holding period is shorter. However, even if Collett et al. (op. cit.) suggest a link between price volatility and holding period, they fail to highlight a proxy for measuring the relationship. For small residential investments, Brown and Geurts (2005) offer an empirical response to the following questions: how long does an investor own

[2] By analogy with the stock market where the question is often selling the stock or keeping it to receive dividends, the question in real estate is selling the property or continue to collect rents.

an apartment building, and why do investors sell some properties more frequently than others do? Using a sample of apartment buildings of between 5 and 20 units over the period 1970 to 1990 in San Diego, California, they found the average holding period to be approximately five years. They conclude that investors sell their assets earlier when values rise faster than rents. Using a microeconomic framework, Brown (2004) shows that consideration of risk that is specific to real estate investments explains why private investors actually own real estate, as well as their buying and selling behaviour, which is more driven by entrepreneurial decision criteria than by financial ones used for other assets. Consistent with this conclusion, applying the Capital Asset Pricing Model (CAPM) for individuals as a way of understanding portfolio management does not lead to relevant results, as demonstrated by Geltner et al. (2006). However, for residential real estate, Cheng et al. (2010) demonstrate that higher illiquidity and transaction costs lead to longer holding periods, while higher return volatility implies shorter holding periods. Regarding the volatility, Rehring (2012) examines the U.K. real estate market and shows that the conditional standard deviation of commercial real estate returns depends on the investment maturity as it is the case for usual stocks in particular on the long-term horizons. The transaction costs and marketing period are also discussed.
These latter results are consistent with previous papers on financial assets.

Several attempts have sought to develop models to determine analytically the optimal holding period for real estate portfolio.[3] Baroni et al. (2007) model the real estate asset through a standard diffusion process, and provide a closed formula for the optimal holding period ex-ante (e.g. for closed funds, when the initial investment is realized). Barthélémy and Prigent (2009) determine the dynamic optimal time to sell using American option approach. Another extension has been proposed by Amédée-Manesme et al. (2015) who incorporate lease structure effect in order to better account for the specificities of real estate. Amédée-Manesme et al. (2016) examine the impact of the market volatility on the optimal holding period of a real estate asset under risk aversion.

Regarding the impact of the taxation on real estate asset, Slack (2000) provides an analysis of the property tax from both the theoretical and practical point of views. Bahl and Martinez-Vazquez (2007) examine the current practice regarding the property tax in developing countries (see also Bird, and Slack, 2004; Bird and Wallace, 2004; Bird, 2007; Slack, 2013).
Slack and Bird (2014) analyze also the political economy of property tax reform. McCluskey and Bell (2008) examine alternative bases for the property tax when considering rental value versus capital value. Nitikin et al. (2012) deals with land taxation in China.

In this paper, our aim is to better emphasize the impact of the taxation on the optimal holding period, extending previous results of Baroni *et al.* (2007). For this purpose, we consider a risk-neutral investor that maximizes the expectation of his global wealth at maturity over a given time period. In this framework, the key parameters are the time horizon and the various taxes such as taxes on the rent and on the capital gains.

[3] Some of the optimization problems are specific to real estate investments and differ from standard financial portfolio management problems (see Karatzas and Shreve, 2001). First, real estate assets exhibit specificities (illiquidity, divisibility, localization etc.). Second, the control variable is the time to sell and not the usual financial portfolio weights as highlighted by Oksendal (2007) for the optimal time to invest in a project with an infinite horizon.

We introduce various taxes and in particular several specific degression functions on the capital gains. Then, we study the behaviour of the optimal time to sell for various financial parameter values and taxation levels. For this purpose, in the standard geometric Brownian framework, we derive a closed-form formula to determine the ex-ante optimal holding period of a real estate asset with taxation. We show in particular that the taxation on the capital gains leads to a special case of real options, namely a Call option with underlying corresponding to the real estate asset price and a strike equal to the spot real estate asset weighted by a coefficient depending potentially on maturity. Our results prove that taxation highly modifies the features of the optimal time to sell the real estate asset. For example, this latter one can jump whereas the expectation of the global wealth is a continuous function with respect to time. We provide numerical illustrations to emphasize such features and to examine the impact of various market and taxation parameters. We also compare the impact of different degression functions on the capital gains for four European countries, namely Germany, England, France and Spain.

The structure of the paper is laid out as follows. Section 2 presents the continuous-time framework and the optimal time to sell with taxation. Numerical results for the optimal holding period are provided in section 3. We analyze several sensitivities of the optimal holding period. We investigate also the impact of different degression functions on the capital gains for four European countries, namely Germany, England, France and Spain. Concluding remarks are gathered in Section 4. A brief survey about taxation is provided in appendix.

2. Continuous-time model and risk neutral investor

In this section, the time of sale is pre-set, committed irrevocably at time 0, based on the expected dynamics of the portfolio value and its cash flow. The real estate portfolio value is defined as the sum of the discounted free cash flows (FCF) and the discounted terminal value (the selling price). We denote k_F and k_P respectively as the discount rate of the free cash flows and of the real estate price, and V_T the terminal value. For a firm, these discount factors can be equal for example to weighted average cost of capital, which corresponds to the firm's cost of capital in which each type of capital is proportionately weighted. For an individual, different numeraires can be used to evaluate the profit to invest. For example, cash flows can be evaluated relatively to an investment in bonds while the real estate asset can be compared to stocks. Another approach can be based on the notion of risk aversion and its representation through utility functions. Barthélémy et al. (2009) show how the choice of discounted rates can be linked to various utility functions, explaining why very high values can be observed in practice.

We assume that the free cash flow grows at a constant rate g [4]. We consider also property taxes on both the cash flows and on the capital gains. For this purpose, let us consider the corresponding tax rates τ_F, τ_P and τ_G. The property tax τ_P is applied to the cash flows. Let f_a be the acquisition fee when buying initially the asset and $f_{cw}(.)$ be the costs of work (these two previous terms are used to determine the capital gains). Additionally, we introduce also a specific function l(.) increasing with respect to time with values in $[0,1]$ to take

[4] This assumption allows explicit solutions for the optimal times to sell and of the optimal portfolio values. The introduction of stochastic rates would lead to only simulated solutions.

account of degression when computing tax on the capital gains.[5] In what follows, we assume that all the previous parameters are deterministic.

2.1. Continuous-time model

Following Baroni *et al.* (2007), we suppose that the price dynamics follows a geometric Brownian motion:

$$\frac{d\tilde{P}_t}{\tilde{P}_t} = \mu dt + \sigma dW_t, \tag{1}$$

where W_t is a standard Brownian motion.

This equation assumes that the real estate return can be modelled as a simple diffusion process where parameters μ and σ are respectively equal to the trend and to the volatility.

Then the future real estate index value at time *t*, discounted at time 0 and with property tax, can be expressed as:

$$P_t = P_0 \exp\left[\left(\mu - k_P - \tfrac{1}{2}\sigma^2\right)t + \sigma W_t\right] \text{ with } E\left[\frac{P_t}{P_0}\right] = \exp\left(\left[\mu - k_P\right]t\right) \tag{2}$$

Denote by FCF_0 the initial value of the free cash flow. The continuous-time version of the sum of the discounted free cash flows FCF_s after tax is equal to:

$$C_t = \int_0^t FCF_0 e^{-[k_F-(g-\tau_F-\tau_P]s} ds = \frac{FCF_0}{k_F+\tau_F+\tau_P-g}\left(1-e^{-[k_F+\tau_F+\tau_P-g]t}\right) \tag{3}$$

Denote $c = \dfrac{FCF_0}{k_F+\tau_F+\tau_P-g}$ and $\alpha = k_F+\tau_F+\tau_P-g.$

Introduce the real estate portfolio value process V after taxes, which is the sum of the discounted free cash flows after taxes and the future real estate index value at time *t*, discounted at time 0 minus the discounted tax on capital gains, namely: [6]

$$V_t = C_t + P_t - \tau_G P_0 . Max\left[\mathrm{e}^{\left[(\mu-0.5\sigma^2)t+\sigma W_t\right]} - (1+f_a+f_{cw}(t)), 0\right](1-l(t))e^{-k_P t}. \tag{4}$$

We determine the portfolio value $V_{\overline{T}}$ for a given maturity $\overline{T}$. This assumption on the time horizon allows accounting of selling constraints before a limit date. The higher $\overline{T}$, the less stringent this limit.

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[5] Such degression corresponds for example to the French case.

[6] For the French case, we have to distinguish two different taxes on capital gains. In that case, we must write: (denote x^+=Max(x,0))

$$V_t = C_t + P_t - \tau_{P,1}(\tilde{P}_t - P_0(1+f_a+f_{cw}(t))^+(1-l_1(t))e^{-k_P t} - \tau_{P,2}(\tilde{P}_t - P_0(1+f_a+f_{cw}(t))^+(1-l_2(t))e^{-k_P t}.$$

Additionally, since 2013, there exists another specific tax on capital gains, namely a piecewise linear function for capital gains above 50000 Euros. This leads to another option value. For simplicity, we keep one term for the general result and use the other terms for the numerical application corresponding to the French case.

2.2. Computation with the linear utility function:

The optimization problem is:

$$Max_{t\in[0,\bar{T}]} E\left[V_t\right]. \tag{5}$$

Note that the expectation of V_t is equal to:

$$E\left[V_t\right] =$$

$$c\left(1-e^{-at}\right)+P_0\,\mathrm{e}^{\left[(\mu-k_P)t\right]}-\tau_G P_0.E\left[Max\left[\mathrm{e}^{\left[(\mu-0.5\sigma^2)t+\sigma W_t\right]}-(1+f_a+f_{cw}(t),0\right]\right](1-l(t))e^{-k_P t}. \tag{6}$$

Denote by N(.) the cumulative distribution function (cdf) of the standard Gaussian probability distribution.

Lemma 1. By adaptating the Black and Scholes formula, we get an explicit formula for the expectation of the capital gains (before applying degression), which looks like a call option C_{BS}:

$$\begin{aligned}
&P_0 E\left[Max\left[\mathrm{e}^{\left[(\mu-0.5\sigma^2)t+\sigma W_t\right]}-(1+f_a+f_{cw}),0\right]\right]=\\
&P_0\left[e^{\mu t}N(d_1)-\left(1+f_a+f_{cw}\right)N(d_2)\right]\\
&with\\
&\begin{cases} d_1=\dfrac{-Log\left(1+f_a+f_{cw}\right)+(\mu+0.5\sigma^2)t}{\sigma\sqrt{t}}\\ d_2=d_1-\sigma\sqrt{t}\end{cases}
\end{aligned} \tag{7}$$

2.2.1 Special case (no tax)

When the investor does not take any tax into account, we get:

$$E\left[V_t\right]=\frac{FCF_0}{k_F-g}\left(1-e^{-[k_F-g]t}\right)+P_0 e^{[\mu-k_P]t}, \tag{8}$$

We determine the optimal solution at time 0, for a given maturity T and for an investor maximizing the expectation of his terminal wealth. We have to analyze the expected utility of the future real estate index value at time t, discounted at time 0: if the price return μ is higher than the discount rate k, then, the optimal solution for the linear utility case is simply equal to the maturity $\bar{T}$. In what follows we assume that $k_P>\mu$ (otherwise the optimal holding period corresponds straightforwardly to the given maturity $\bar{T}$). We also assume that $\mu-g+\left(k_F-k_P\right)>0$. Note that when $k_F=k_P$, this condition is equivalent to $g<\mu$, condition which is usually empirically satisfied. Consequently, not selling the asset implies higher cumulated cash but a smaller discounted expected terminal value $\mathrm{P}_0\mathrm{e}^{(\mu-\mathrm{k_P})\mathrm{t}}$. Hence, the investor has to choose between more (discounted) flows and less expected discounted index value.

Then, the optimal holding period is determined as follows (see Barthélémy and Prigent, 2009, for the case $k_F = k_P$).

- *1: The initial price* P_0 *is smaller than* $\frac{FCF_0}{k_P-\mu} e^{-(\mu-g+(k_F-k_P))\overline{T}}$.

Then, the optimal time to sell T^* corresponds to the maturity $\overline{T}$. Since *the Price Earning Ratio* (PER)[7] is too small ($< \frac{e^{-(\mu-g+(k_F-k_P))\overline{T}}}{k_P-\mu}$), the sell is not relevant before maturity.

- *2: The initial price* P_0 *lies between the two values* $\frac{FCF_0}{k_P-\mu} e^{-(\mu-g+(k_F-k_P))\overline{T}}$ and $\frac{FCF_0}{k_P-\mu}$.

The optimal time to sell T^* is solution of the following equation:

$$\frac{\partial E\left[V_t\right]}{\partial t} = 0, \tag{9}$$

which solves to (see Baroni et al., 2007 for the case $k_F = k_P$):

$$T^* = \frac{1}{\left[\left(\mu - g\right) + \left(k_F - k_P\right)\right]} \ln\left(\frac{FCF_0}{P_0} \times \frac{1}{k_P - \mu}\right). \tag{10}$$

In particular, note that T^* is a decreasing function of the initial price P_0 and of the difference between the index return μ and the growth rate g of the free cash flows. This latter property was empirically observed by Brown and Geurts (2005). It means that investors sell property sooner when values rise faster than rent.

- *3: The initial price* P_0 *is higher than* $\frac{FCF_0}{k_P-\mu}$.

The optimal time to sell T^* corresponds to the initial time 0. Since the PER $\frac{P_0}{FCF_0}$ is sufficiently large ($> \frac{1}{k_P-\mu}$), there is no reason to keep the asset P.

We note that the discounted expected value V_t of the portfolio is concave. Knowing the optimal time to sell T^* which is deterministic, the probability distribution of the discounted portfolio value V_{T^*} can be determined. The value V_{T^*} is equal to:

$$V_{T^*} = \frac{FCF_0}{(k_F - g)}\left(1 - e^{-(k_F - g)T^*}\right) + P_0 \exp\left[\left(\mu - k_P - 1/2\sigma^2\right)T^* + \sigma W_{T^*}\right].$$

Denote $A = \frac{FCF_0}{(k_F-g)}\left(1 - e^{-(k_F-g)T^*}\right)$ the cumulative discounted free cash flow value at T^*. Since, from (10), the optimal time to sell satisfies:

$$T^* = \frac{1}{\left[\left(\mu - g\right) + \left(k_F - k_P\right)\right]} \ln\left(\frac{FCF_0}{P_0} \times \frac{1}{k_P - \mu}\right),$$

then, we deduce:

[7] The PER $\frac{P_0}{FCF_0}$ is defined as the initial price of a property or group of properties divided by the free cash flows produced by this or these properties. It measures the current price relative to its cash flows production. The PER is often known in real estate as the price multiple.

$$A = \frac{FCF_0}{(k_F - g)} \left(1 - \left[\frac{FCF_0}{P_0 (k_P - \mu)} \right]^{\left(-\frac{k_F - g}{\mu - g + (k_F - k_P)} \right)} \right),$$

and the cumulative distribution function of the terminal wealth $F_{V_{T^*}}$ of V_{T^*} is given by:

$$F_{V_{T^*}}(v) = \begin{cases} 0, \text{if } v \leq A \\ N\left[\frac{1}{\sigma\sqrt{T^*}} \left(\ln\left(\frac{v - A}{P_0} \right) - \left(\left(\mu - k_P - 1/2\sigma^2 \right) T^* \right) \right) \right], \text{if } v > A \end{cases} \tag{11}$$

where N denotes the cdf of the standard Gaussian distribution**.**

2.2.2 Special case (taxation but not on capital gains)

In this subsection, our aim is to examine the impact of taxation on the optimal holding period for risk-neutral investor. In particular, we study the influence of the degression condition on the capital gains.

For this purpose, let us introduce the function defined by:

$$h(t, \tau_F, \tau_P) = \frac{FCF_0}{k_F + \tau_F + \tau_P - g} \left(1 - e^{-[k_F + \tau_F + \tau_P - g]t} \right) + P_0 e^{[\mu - k_P]t}, \tag{12}$$

This function gives the expected value of the portfolio when there is no taxation on the capital gains. In what follows, we assume that $k_P > \mu$ (otherwise the optimal holding period corresponds straightforwardly to the given maturity $\overline{T}$).

Introduce now the optimal holding period $T^*(\tau_F, \tau_P, \tau_G)$ with taxation on the capital gains. Assume that $(\mu - g) + (\tau_F + \tau_P) + (k_F - k_P) > 0$. We also assume that the initial price P_0 lies between the two values $\frac{FCF_0}{k_P + \tau_P - \mu} e^{-(\mu - g + (k_F - k_P) + (\tau_F + \tau_P))\overline{T}}$ and $\frac{FCF_0}{k_P - \mu}$.

Proposition 1 (no taxation on capital gains). When there is no taxation on the capital gains, the optimal holding period with taxation on the rents is smaller than the optimal holding period with no taxation on the rents.

Proof. We have:

$$T^*(\tau_F, \tau_P, 0) = \frac{1}{\left[(\mu - g) + (\tau_F + \tau_P) + (k_F - k_P) \right]} \ln\left(\frac{FCF_0}{P_0} \times \frac{1}{k_P - \mu} \right) \tag{13}$$

Thus $T^*(\tau_F, \tau_P, 0)$ is decreasing with respect to τ_F and τ_P for usual parameter values. It mains that the higher the taxes on cash flow, the quicker the investor will try to benefit from real asset prices increase.

2.2.3 General case (taking all taxes into account)

Denote by $C_{BS}(t)$ the expectation $e^{-\mu_P t}E\left[Max\left[e^{\left[(\mu_P-0.5\sigma^2)t+\sigma W_t\right]}-(1+f_a+f_{cw}),0\right]\right]$ which corresponds to the value of the call option with spot underlying value equal to 1 and strike equal to $(1+f_a+f_{cw})$ and riskless rate equal to μ_P.

Proposition 2 (general case). Assuming that the expectation of the capital gains (after applying the degression) is decreasing with respect to time and that the function $h(t,\tau_F,\tau_P)$ is convex with respect to t, then the optimal holding period with taxation on the capital gains is higher than the optimal holding period with no taxation on the capital gains.

Proof.

By definition, the holding period $T^*(\tau_F,\tau_P,0)$ maximizes $h(t,\tau_F,\tau_P)$ while the holding period $T^*(\tau_F,\tau_P,\tau_G)$ maximizes:

$$h(t,\tau_F,\tau_P)-\tau_G P_0 C_{BS}(t)-(1-l(t))e^{-k_P t}.$$

By differentiating, we get:

$$\frac{\partial h(T^*(\tau_F,\tau_P,0),\tau_F)}{\partial t}=0$$

and

$$\frac{\partial h(T^*(\tau_F,\tau_P,\tau_G),\tau_F,\tau_P)}{\partial t}=$$

$$\frac{\partial}{\partial t}\left[\tau_G P_0 C_{BS}(t)(1-l(t))e^{-k_P t}\right]$$

By assumption on the expectation of the capital gains, the last term of previous equation is negative. Now, by assumption on the function $h(t,\tau_F,\tau_P)$, $\frac{\partial h(t,\tau_F,\tau_P)}{\partial t}$ is decreasing. Thus, since $\frac{\partial h(T^*(\tau_F,\tau_P,0),\tau_F,\tau_P)}{\partial t}=0$, we deduce that $T^*(\tau_F,\tau_P,0)<T^*(\tau_F,\tau_P,\tau_G)$.

Lemma 2. Using the formula corresponding to the sensitivity of the call option C_{BS} to time to maturity (the so called Greek Theta), we have:

$$\Theta(t)=\frac{\partial}{\partial t}C_{BS}(t)=\frac{N'(d_1)\sigma}{2\sqrt{t}}-\mu(1+f_a+f_{cw})e^{-\mu t}N(d_2)>0. \quad (14)$$

Proposition 3. The expectation of the capital gains after applying degression is decreasing with respect to time as soon as we have: $\frac{\Theta(t)}{C_{BS}(t)}<(k_P-\mu)+\frac{l'}{1-l}$.

Proof.

We have:

$$\frac{\partial}{\partial t}\left(E\left[Max\left[\mathrm{e}^{\left[(\mu-0.5\sigma^2)t+\sigma W_t\right]}-(1+f_a+f_{cw}),0\right]\right](1-l(t))e^{-k_P t}\right)=$$

$$\frac{\partial}{\partial t}\left[C_{BS}\left(t\right)(1-l(t))e^{-(k_P-\mu)t}\right].$$

Since the function l(.) is assumed to be increasing with values in [0,1] and $k_P > \mu$, the function $(1-l(t))e^{-(k_P-\mu)t}$ is decreasing. Denote m(t) this latter function.
Searching for a zero of equation

$$\frac{\partial}{\partial t}E\left[Max\left[\mathrm{e}^{\left[(\mu-0.5\sigma^2)t+\sigma W_t\right]}-(1+f_a+f_{cw}),0\right]\right](1-l(t))e^{-k_P t}=0$$

We get the following condition:

$$\frac{-\Theta(t)}{C_{BS}(t)}=-\frac{m'(t)}{m(t)}=\left(k_P-\mu\right)+\frac{l'}{1-l}.$$

Therefore, we deduce that the expectation of the capital gains is decreasing as soon as:

$$\frac{-\Theta(t)}{C_{BS}(t)}<\left(k_P-\mu\right)+\frac{l'}{1-l}. \tag{15}$$

Remark 1. For standard parameter values and degression functions l(.), the previous condition is not always satisfied, leading either to local maxima or local minima (see Section 3 for specific numerical illustrations).

In what follows, we examine the sensitivity of the expectation of the portfolio value with respect to the trend of the real asset.

Proposition 4. The expectation of the portfolio value is increasing with respect to the trend of the real asset as soon as it is sufficiently high and/or the time horizon is sufficiently high.

Proof. Using the formula corresponding to the sensitivity of the call option C_{BS} to interest rate (the so called Greek Rho), we have: (here μ corresponds to the interest rate used in the Black-Scholes formula)

$$Rho(t)=\frac{\partial}{\partial\mu_P}C_{BS}\left(t\right)=tN(d_2)e^{-\mu_P t}(1+f_a+f_{cw})>0.$$

Therefore, we deduce that:

$$Rho(\bar{V}_t)=P_0te^{-k_P t}\left[e^{\mu_P t}-\tau_G N(d_2)(1+f_a+f_{cw})(1-l(t))\right]. \tag{16}$$

Since both $N(d_2)$ and $(1-l(t))$ are smaller than 1, $Rho(\bar{V}_t)$ is positive if $e^{\mu_P t}\succ\tau_G(1+f_a+f_{cw})$.

In what follows, we examine the sensitivity of the expectation of the portfolio value with respect to the volatility of the real asset. Recall that, when there is no taxation on the capital gains, this expectation does not depend on the volatility.

Proposition 5. The expectation of the portfolio value is decreasing with respect to the volatility of the real asset.

Proof. Using the formula corresponding to the sensitivity of the call option C_{BS} to volatility (the so called Greek Vega), we have:

$$Vega(t) = \frac{\partial}{\partial \sigma} C_{BS}\left(t\right) = N'(d_1)\sqrt{t} > 0.$$

Therefore, we deduce that:

$$Vega(\bar{V}_t) = -\tau_G P_0 N'(d_1)\sqrt{t}(1-l(t))e^{-k_P t} \prec 0.$$

Remark 1. The sensitivities of the optimal holding period can be analytically analysed using the implicit functions theorems applied to the derivatives of the expectation of the global wealth which must be null. However, it leads to rather complicated formulas. In Section 3, we use the graphics of the expectation of the global wealth to examine how the optimal to sell behaves.

3. Numerical analysis

3.1. The numerical base cases

We investigate two main numerical cases. These cases are those of Baroni et al. (2007):

- **Case 1** corresponds to an early selling, due in particular to weak expected return of the real estate asset. We set:

  $$\mu = 4.4\%, \sigma = 5\%, g = 3\%, k = 8.4\%, P_0 = 100, FCF_0 = 100/22.$$

  When there is no taxation, we get the following function describing how the expected value evolves along time (see Figure 1).

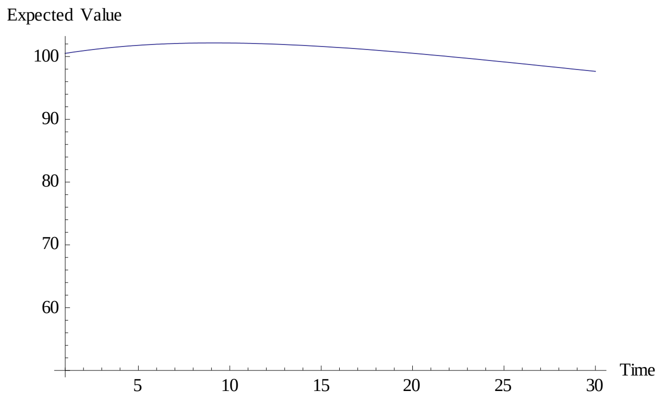


Figure 1: Expected value with no taxation (early selling)

  The optimal time to sell is T*=9 years.

- **Case 2** corresponds to a late selling, due in particular to higher expected return of the real estate asset. We set:

  $$\mu = 6\%, \sigma = 5\%, g = 2\%, k = 9.5\%, P_0 = 100, FCF_0 = 100/15.$$

  When there is no taxation, we get the following function describing how the expected value evolves along time (see Figure 2) .

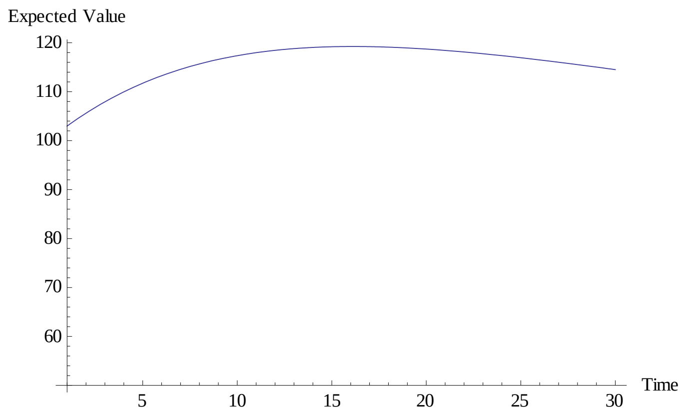


Figure 2: Expected value with no taxation (late selling)

The optimal time to sell is T*= 16 years.

3.2. The optimal holding with taxation (base cases)

Now we take account of the taxation. Our numerical base case is as follows: The tax on the cash flows τ_F is equal to 30%. The property tax τ_P applied to the cash flows is equal to 8%. The tax on the capital gains τ_G is equal to 30%. The acquisition fee f_a when buying initially the asset is equal to 5%. Finally the costs of work f_{cw} (.) is equal to 15%.

We investigate two kinds of degression function: the first one, which is used for example in Germany, Spain and England, corresponds to a piecewise constant functions, namely lc(.) is equal to 0 on the time interval $[0,10]$ and equal to 1 for maturity longer than 10 years (it means that the taxation on the capital gains is null after ten years); the second one, which a simplified version of the French case, corresponds to a piecewise linear functions, namely li(.) is equal to a piecewise linear function on the time interval $[0,22]$ (it means that the taxation on the capital gains is linearly decreasing on the first 22 years) and equal to 1 for maturity longer than 22 years (it means that the taxation on the capital gains is null after 22 years).

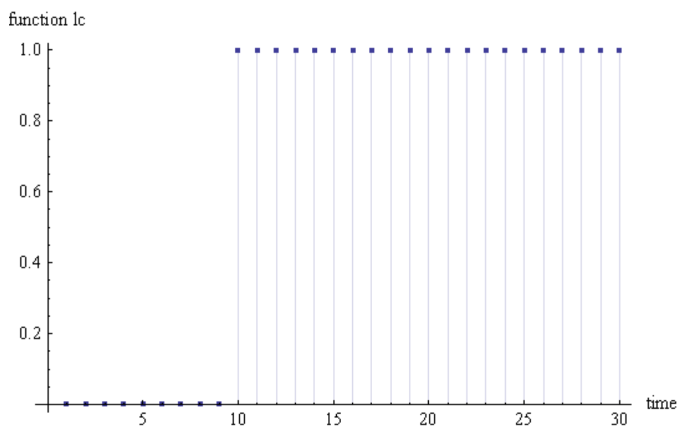


Figure 3. Degression function lc

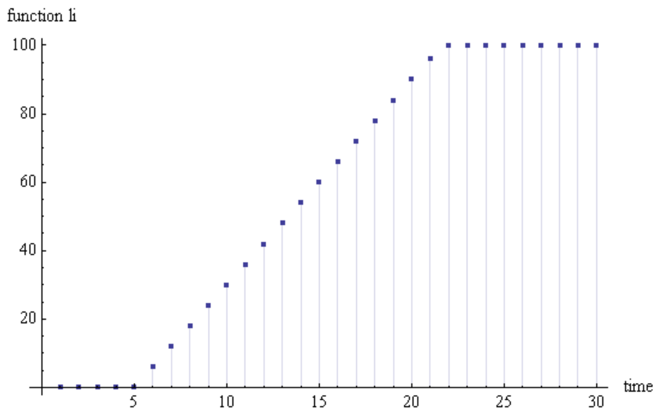


Figure 4. Degression function li

3.2.1. Piecewise constant degression (lc case)

For case 1, we get:

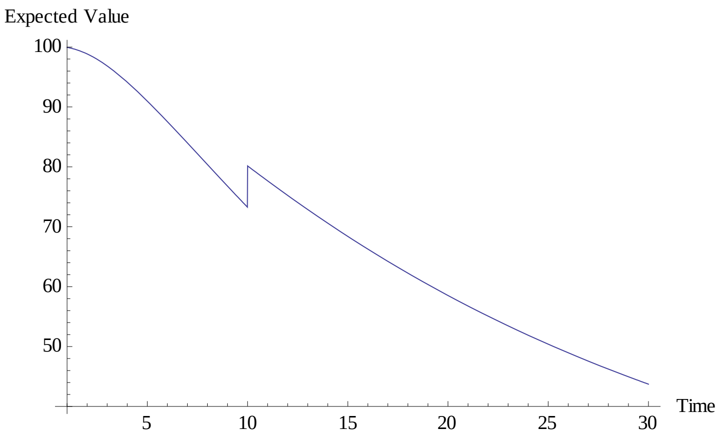


Figure 5: Expected value with taxation (early selling)

We note that the impact of taxation corresponds to immediately sell the real estate asset.

For case 2, we get:

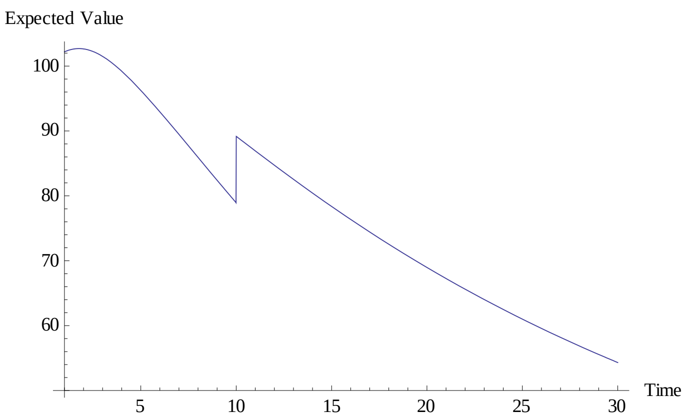


Figure 6: Expected value with taxation (late selling)

In that case, the taxation implies to sell more quickly the real estate asset (here, T*= 1.5 years).

As it can be seen in all the cases, the behaviour of the expected value is quite significantly modified when there is taxation.

3.2.2. Piecewise linear degression (li case)

When the degression function is piecewise linear, we get similar properties except that the expected value is a continuous function with respect to maturity.

For case 1, we get:

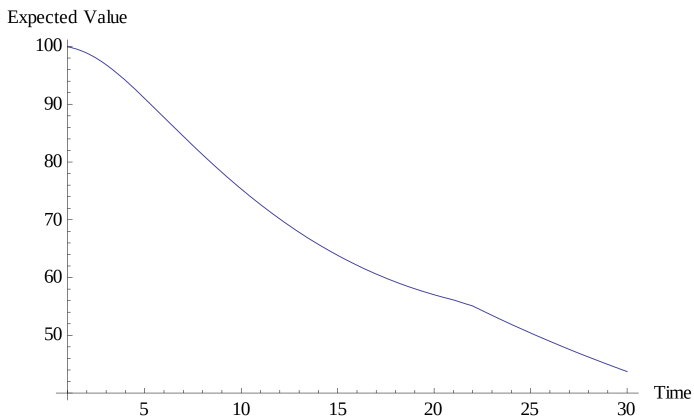


Figure 7: Expected value with taxation (early selling)

We note that the impact of taxation corresponds to immediately sell the real estate asset.

For case 2, we get:

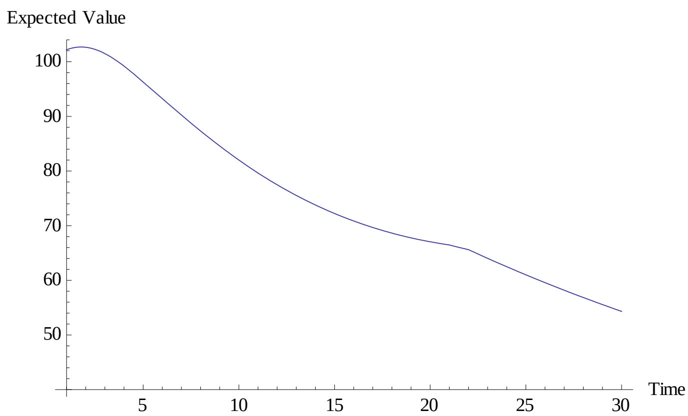


Figure 8: Expected value with taxation (late selling)

In that case, the taxation implies to sell more quickly the real estate asset (here, T*= 2 years).

3.3. Sensitivity to the PER ratio

In this subsection, we analyse the impact of the PER ratio $\frac{P_0}{FCF_0}$ in the taxation framework.

3.3.1. Piecewise constant degression (lc case)

Consider the degression function lc. For case 1, we get the following features illustrated in Figure 9.

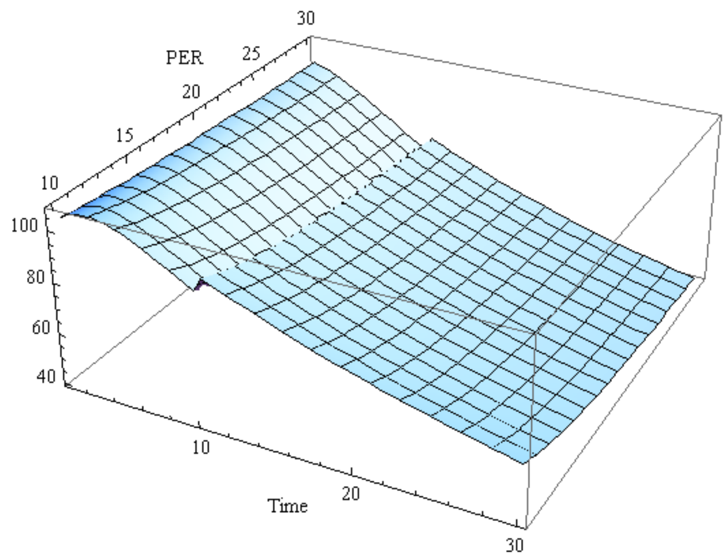


Figure 9: Expected value with respect to time and PER (case 1, function lc)

We note that both the expected value and the optimal time to sell is not deeply modified when the PER is varying. Note also that the expected value is not a monotone function with respect to the PER.

For case 2, we get similar features as illustrated in Figure 10.

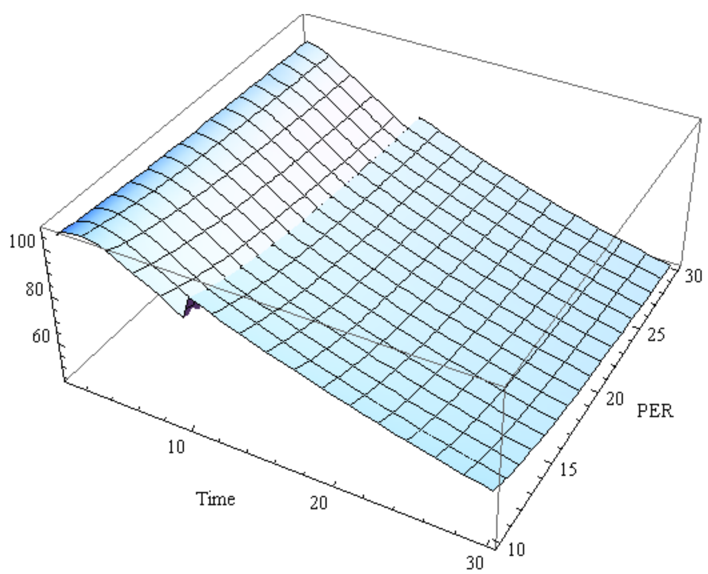


Figure 10: Expected value with respect to time and PER (case 2, function lc)

3.3.2. Piecewise linear degression (li case)

When the degression function is piecewise linear, we get similar properties except that the expected value is a continuous function with respect to maturity.

Consider the degression function li. For case 1, we get the following features illustrated in Figure 11.

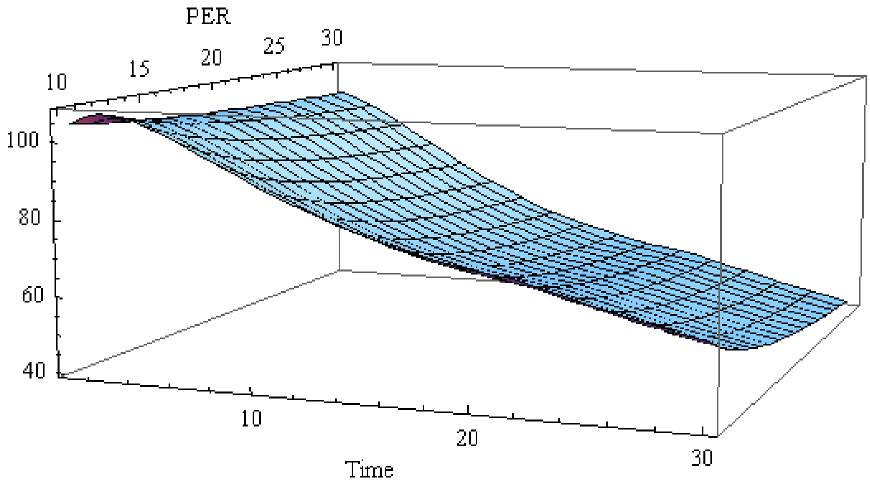


Figure 11: Expected value with respect to time and PER (case 1, function li)

For case 2, we get similar features as illustrated in Figure 12.

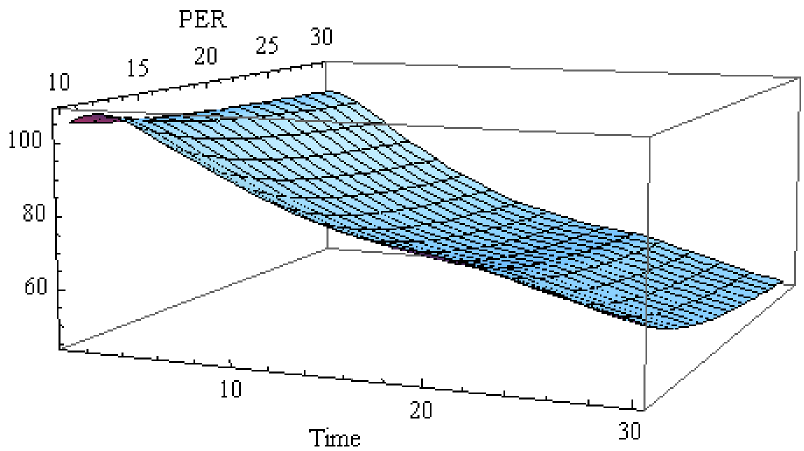


Figure 12: Expected value with respect to time and PER (case 2, function li)

3.4. Sensitivity to the trend of the real asset price

In what follows, we start from the numerical base case corresponding to case 1 and we consider various trend values of the real asset price.

3.4.1. Piecewise constant degression (lc case)

Figure 13 illustrates the behaviour of the expected value. The lower curve corresponds to $\mu = 4.4/2\%$ and the upper curve to $\mu = 4.4 \times 2\%$. The other cases correspond (from the bottom) respectively to $\mu = 4.4\%, 4.4 \times 1.5\%, 4.4 \times 1.7\%$ and $4.4 \times 1.9\%$. First, we note that the expected value is an increasing function of the trend of the real estate price. Second, according to intuition, the optimal time to sell is also increasing with respect to the trend. An interesting feature is that the optimal time to sell the real estate asset can suddenly jump for example from about 3 to about 10 years (see for instance the blue curve corresponding to $\mu = 4.4 \times 1.7\%$). This is due to the digression of the taxation on the capital gains. For the higher trend value, it would be optimal to sell at the upper bound on the maturity, namely 30 years for this example.

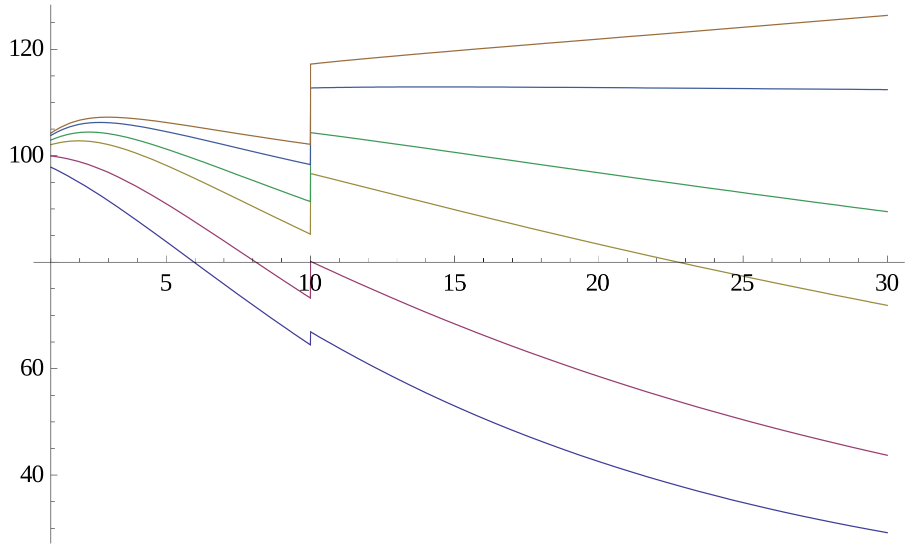


Figure 13: Expected value with respect to real estate asset trend value (lc)

Figure 14 displays the expected value for all trends values lying from $\mu = 4.4/2\%$ to $\mu = 4.4 \times 2\%$.

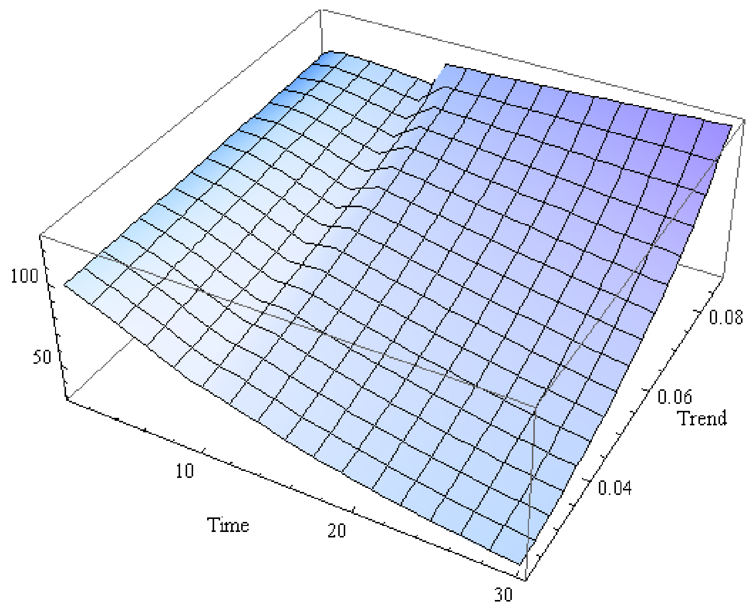


Figure 14: Expected value with respect to real estate asset trend value (lc)

3.4.2. Piecewise linear degression (li case)

As previously, the expected value is an increasing function of the trend of the real estate price. Second, the optimal time to sell is also increasing with respect to the trend. Despite the continuity of the expected value with respect to time, the optimal time to sell the real estate asset can still suddenly jump for example from about 3 to about 22 years (see for instance blue curve corresponding to $\mu = 4.4 \times 1.7\%$). For this latter case, the time at which there is no longer taxation on the capital gains is optimal, while, for higher values of the trend, the function is strictly increasing after this specific time. Note also that there exist local minima, as suggested in Remark 1.

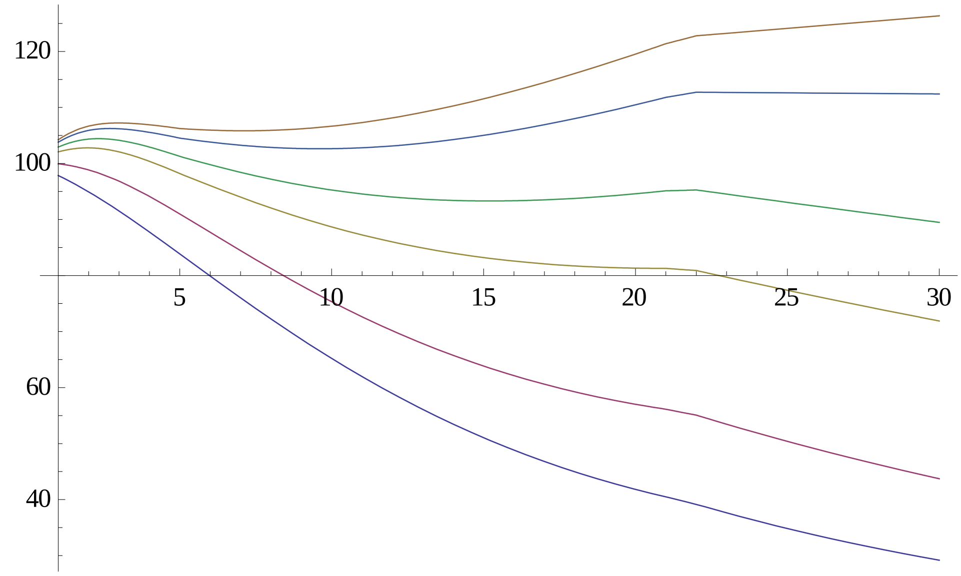


Figure 15: Expected value with respect to real estate asset trend value (li)

Figure 16 displays the expected value for all trends values lying from $\mu = 4.4/2\%$ to $\mu = 4.4 \times 2\%$.

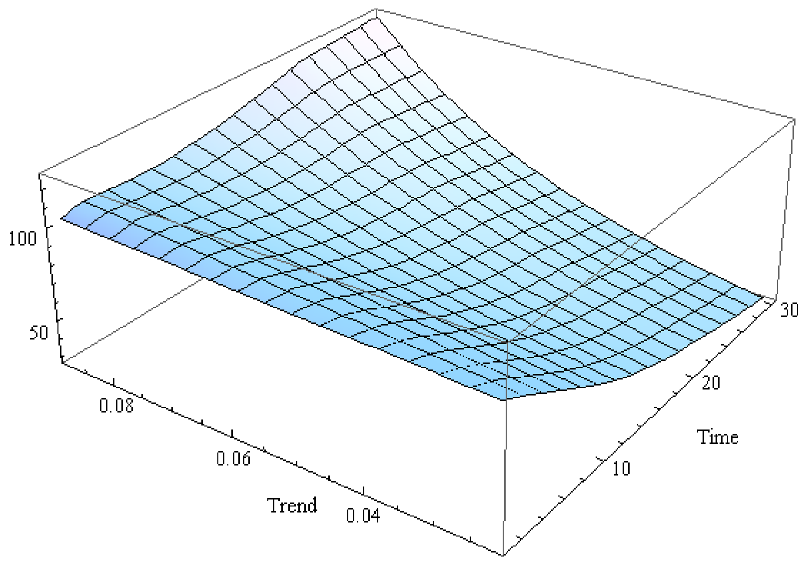


Figure 16: Expected value with respect to real estate asset trend value (li)

3.5. Sensitivity to the tax on the capital gains

We investigate two cases depending on the level of the trend: a medium trend and a high trend. These two case are illustrated by Figures 17 and 18. For the value of the tax τ_G on the capital gains, we investigate six cases defined from the base case corresponding to $\tau_G = 30\%$, namely $15\%, 24\%, 30\%, 39\%, 48\%$ and 60%.

3.5.1. Piecewise constant degression (lc case)

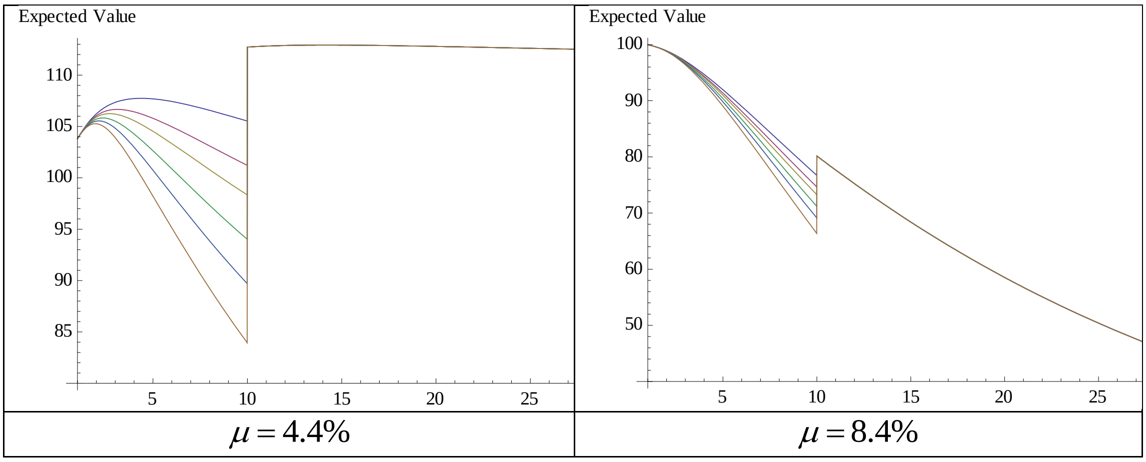


Figure 17: Expected value with respect to tax on the capital gains (li)

The expected value is obviously decreasing with respect to the tax on the capital gains. If the time horizon is smaller than 10 years, the optimal time to sell is (slightly) decreasing with respect to tax τ_G when the trend of the real estate asset is not too high.

3.5.2. Piecewise linear degression (li case)

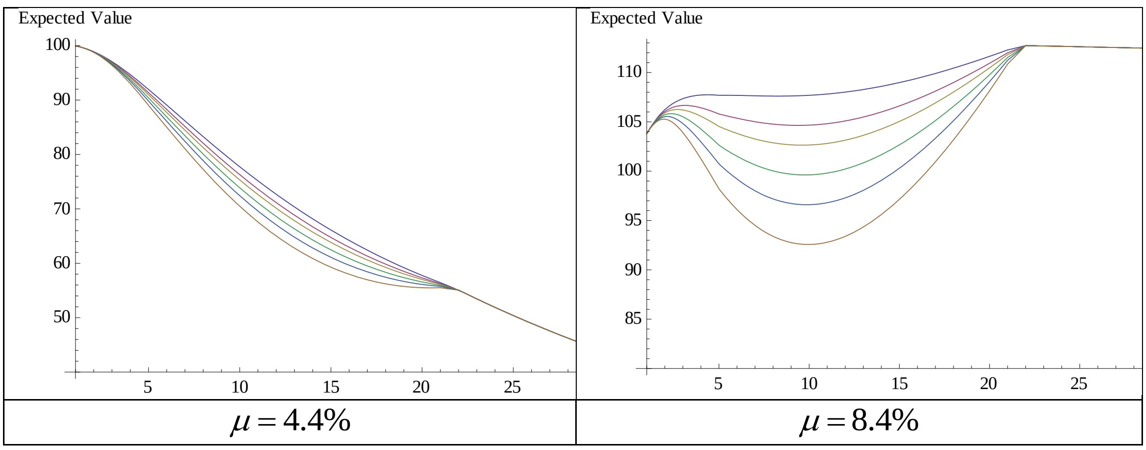


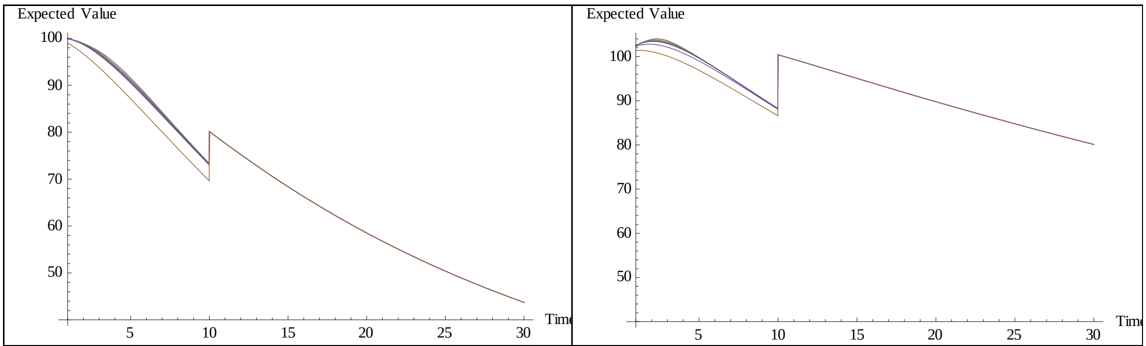
Figure 18: Expected value with respect to tax on the capital gains (li)

If the time horizon is smaller than 10 years, the optimal time to sell is (slightly) decreasing with respect to tax τ_G when the trend of the real estate asset is sufficiently high.

3.6. Sensitivity to the real estate asset volatility

We start from the numerical base case 1. For the value of the volatility, we investigate six cases defined from the base case corresponding to $\sigma = 5\%$, namely $2.5\%, 4\%, 5\%, 6\%, 10\%$ and 20%. As proved in Proposition 5, the expectation of the portfolio value is decreasing with respect to the volatility of the real asset. Figures 19 and 20 show that the impact of the volatility is not very significant for small trend values and/or standard real estate volatility levels. This is due to the fact that we consider here risk-neutral investors. Recall that, for this case, the volatility does not play any role when taxation is not taken into account.

3.6.1. Piecewise constant degression (lc case)

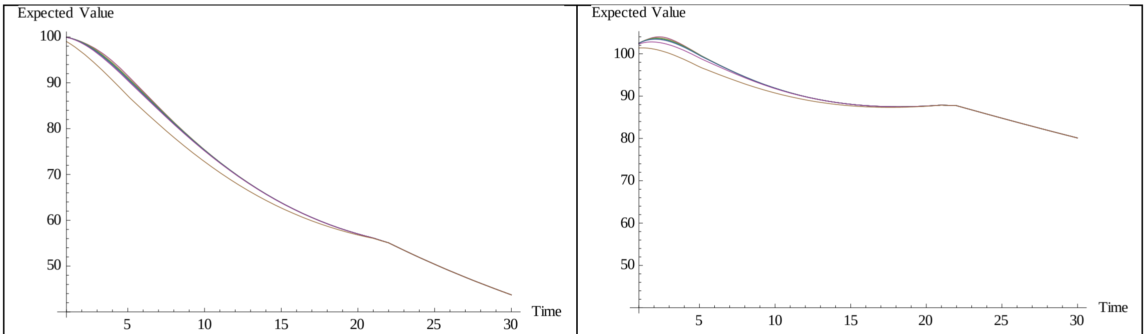


$\mu = 4.4\%$	$\mu = 7\%$

Figure 19: Expected value with respect to the volatility (li)

The expected value is obviously decreasing with respect to the volatility. If the time horizon is smaller than 10 years, the optimal time to sell is (sligtly) decreasing with respect to the volatility when the trend of the real estate asset is not too high.

3.6.2. Piecewise linear degression (li case)



$\mu = 4.4\%$	$\mu = 7\%$

Figure 20: Expected value with respect to the volatility (li)

3.7. Comparison of the li case and the lc case

For this comparison, since the key market parameter is the trend of the real estate asset, we investigate three cases depending on the level of the trend: a low trend, medium trend and a high trend.

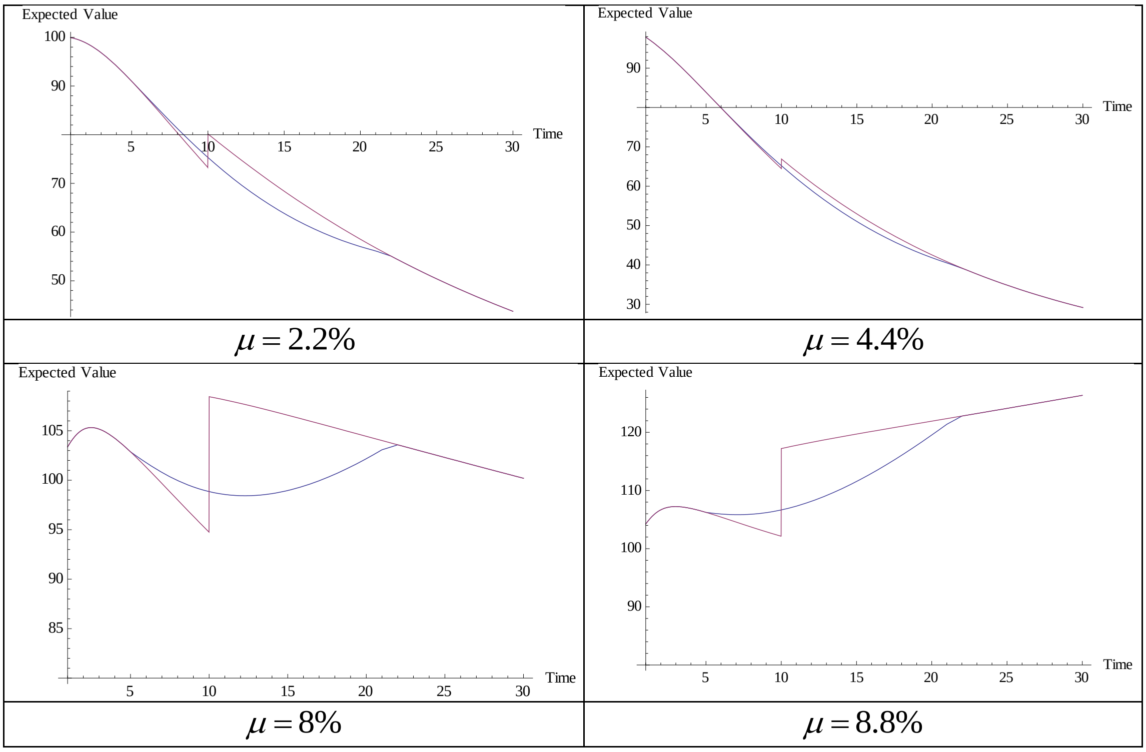


Figure 21: Comparison of the impact of the two digression functions

As expected, the difference is significant for high values of the trend. In such a case, the difference between the two times at which there is no longer taxation on the capital gains has a strong influence on the expectation of the global wealth.

For this numerical example, when $\mu = 8\%$, we note that, if the terminal date is smaller than 10 years, the optimal holding period are equal but, for longer maturities, they highly differ. Indeed, for the lc case, the optimal holding period is equal to the time at which the taxation on the capital gains does no longer hold (here 10 years) while, for the li case, it corresponds to an early selling (about 3 years) and not at all to the time at which there is no more taxation on the capital gains (here 22 years).

4. Conclusion

This paper emphasizes how the taxation on the real estate market impacts the optimal holding period, in particular for relatively high trends of the real estate asset. For this purpose, the investor is assumed to be risk-neutral, which is an usual assumption when dealing with standard real estate literature. Introducing two standard degressive taxations on the capital gains, we show that the corresponding optimal times to sell may highly differ. Note also that, despite the continuity of the expected value with respect to time for one of these digressive functions, the optimal time to sell the real estate asset can still suddenly jump. Various extensions can be introduced. First, as emphasized by Barthélémy and Prigent (2009), it would be interesting to determine the dynamic optimal holding period using the American option approach. In such a case, the fund is no longer closed (i.e. the optimal date is computed once for all at the initial time). In the quasi linear case (risk-neutral investor), we shall expect to get explicit conditions on the real estate price to determine the dynamic optimal time to sell, using previous results. Second, to better take account of the real estate volatility, it would be useful to introduce the notion of risk-aversion as illustrated by Amédée-Manesme et al. (2016) when there is no taxation. Another issue could be to examine the role of the financing of property, since here we have implicitly assumed that either the property was already paid, either we were in the Modigliani-Miller framework as regards the active/passive management. Of course, since all processes of interest are stochastic, it would be interesting to examine how taking account of this feature significantly modify or not previous results (but only simulations could be used). Finally, as shown in Amédée-Manesme et al. (2016), we could introduce the notion of compensating variation to compare several types of holding periods of the three optimal strategies, including in particular the buy-and hold investment strategy.

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Appendix

A.1. The case of Spain

Surplus value payable by tax resident sellers in Spain is 21% of net profit (same as for non-residents). The tax residents IBI ("Impuesto sobre bienes inmuebles") is paid annually or quarterly. Its amount is calculated by the Mayor of each municipality and depends on the cadastral value of housing and municipal tax rate. For example, the average of IBI paid in a city like Barcelona amounted to € 700 per year in 2015. The surplus value is payable by tax resident sellers in Spain is 21% of net profit (19% for non-residents from 2016).

A.2. The case of England

Most financial and capital gains for UK taxpayers are taxed at a fixed rate depending on the resident's tax bracket. So if his income is less than £ 35,000, the rate is 18%, and if it is above £ 35,000, the rate is 28%. However, every taxpayer has an annual allowance of £ 10,000. Most of the gains on asset disposals undergo this tax. The tax on the capital gain is due for individuals but also for companies. For individuals, there is an exemption of £ 10,100 per year. The rate is 18% for taxpayers in the slice to 20%, but 28% to taxpayers slices 40% and 50%. It is not however charged for sales of primary residence.

A.3. The case of Germany

Any real estate owned in Germany is generally taxable in the country. The main residence is not taxable outside the municipal taxes property tax type ("Grundsteuer"). The taxation of the letting of property is composed of property income ("Einkünfte aus und Vermietung Verpachtung") calculated by deducting the expenses. For example, we can integrate the amortization of purchased building (2% of the purchase price excluding land value for the property dates back to 1926 and after, 2.5% of the price for previous property) and the amortization of heavy work or materials on an appropriate period of time. The calculated income is integrated into the overall return. For the taxation of capital gains on sale of property, the capital gains on a principal residence are taxed if the property was inhabited the year of sale and the previous 2 years. Capital gains on other property are not taxable after 10 years of detention. For a shorter period of detention of the property, the capital gain is incorporated in income in the year the property is sold. Concerning the taxation on real estate gain, if the property is sold after 10 years of detention, the capital gain is not taxed. If the property is sold before 10 years of detention, the amount of tax on the capital gain is calculated from the income tax as follows:

Table 1: German Income Tax Rate

up to € 8,004	0%
8 005-13 469 €	From 14 to 24%
13 470-52 881 €	From 24 to 42%
52 882-250 730 €	42%
higher than € 250 730	45%

A. 4. The case of France

In calculating the cost of purchasing must take into account the housing purchase price of acquisition costs (legal fees and any costs of real estate agency) at either a real package 7.5%

of the price of the property, taking into account your cost of work (expansion, renovation, improvement, etc.) or a flat fee of 15% of the purchase price after five years of detention. The principle of allowances for holding period is more than long you keep your home between the time of purchase and that of its sales and more you get a reduction in your capital gain brute. The scale in force in 2016 for allowances is this: year holding period for full abatement tax abatement of 19% for CSG CRDS 15.5%. The purchase price refers to the price at which the seller had purchased the property and includes expenses and allowances paid to the owner at the time of purchase jointly with acquisition costs: registration fees or VAT paid at the time of acquisition, notary fees ... If the seller cannot find the evidence, it can apply a similar flat-rate increase to 7.5% of the purchase price. Increased expenditures in the context of work (construction, expansion, improvement…) are taken into account provided the amount can be justified. If not, the person transferring the property may still increase the purchase price of 15% if, and only if, it owns over 5 years.

Taxation on capital gains are degressive. Additionally, there are actually two scales as regards the imposition of 19% and the other concerning social contributions 15.5% (CSG and CRDS). The first allows an exemption complete after 22 years while for the second one it takes 30 years to not pay any social contribution. Since 1 February 2012, for income tax, the allowance for holding period applies as follows: 6% for each year of detention beyond the 5th until the 21st; 4% for the past 22 years in detention

.

Table 2: French real estate income tax

L'abattement pour durée de détention pour l'impôt sur le revenu	
Durée de détention	Abattement
Source : Direction générale des Impôts	
Entre 0 et 5 ans	0%
6 ans	6%
7 ans	12%
8 ans	18%
9 ans	24%
10 ans	30%
11 ans	36%
12 ans	42%
13 ans	48%
14 ans	54%
15 ans	60%
16 ans	66%
17 ans	72%
18 ans	78%
19 ans	84%
20 ans	90%
21 ans	96%
22 ans	100%

For social deductions, the deduction for holding period is as follows: 1.65% per year of detention beyond the 5th until the 21st; 1.60% for the 22nd year in detention; 9% for each year beyond the 22nd.

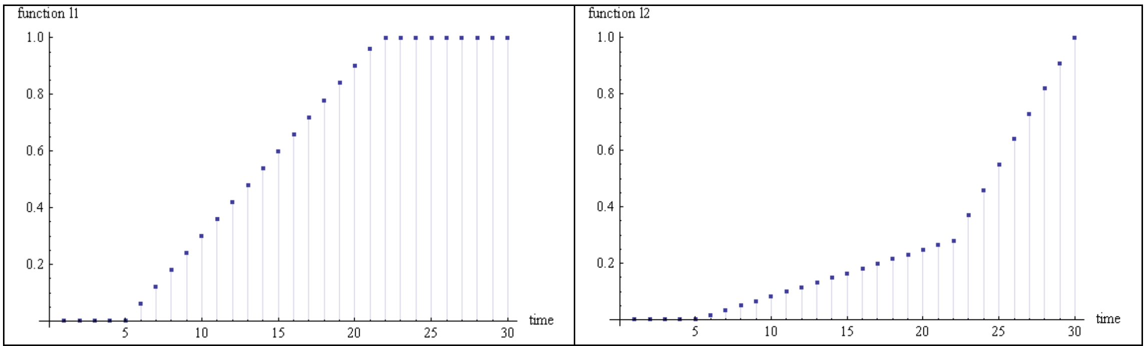
Table 3: French social security contributions

L'abattement pour durée de détention pour les prélèvements sociaux	
Délai de détention	Pourcentage d'abattement
Source : Direction générale des impôts	
Jusqu'à 5 ans de détention	0%
6e année	1,65%
7e année	3,30%
8e année	4,95%
9e année	6,60%
10e année	8,25%
11e année	9,90%
12e année	11,55%
13e année	13,20%
14e année	14,85%
15e année	16,50%
16e année	18,15%
17e année	19,80%
18e année	21,45%
19e année	23,10%
20e année	24,75%
21e année	26,40%
22e année	28%
23e année	37%
24e année	46%
25e année	55%
26e année	64%
27e année	73%
28e année	82%
29e année	91%
30e année	100%

As seen previously, for the French case, we have to distinguish two different taxes on capital gains. In that case, we must consider two digressive functions to take account of the two scales when computing the global wealth. (recall the notation x+=Max(x,0))

$$V_t = C_t + P_t - \tau_{P,1}(\tilde{P}_t - P_0(1+f_a+f_{cw}(t))^+(1-l_1(t))e^{-k_P t} - \tau_{P,2}(\tilde{P}_t - P_0(1+f_a+f_{cw}(t))^+(1-l_2(t))e^{-k_P t}.$$

We get the following graphs:



Digressive function for income tax	Digressive function for social security contributions

Figure 22: The two French digressive functions

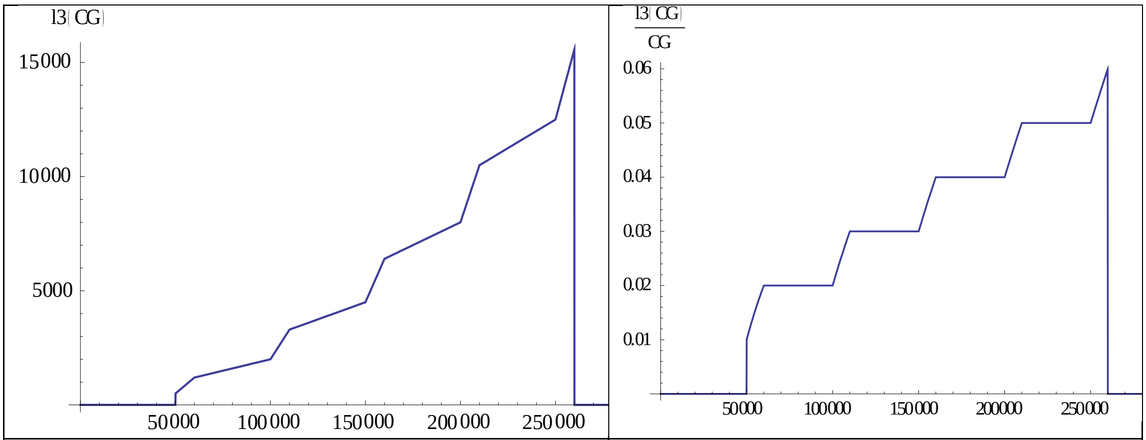
A surcharge for the real estate gains exceeding € 50,000

Since 2013, the government has introduced a surcharge on the capital gains of more than 50,000 €. It aims to impose more heavily those who make a significant real estate gain. This surcharge is still relevant in 2016. The rate of the surcharge varies from 0-6% depending on the magnitude of the gain. The maximum tax is therefore 25% (excluding social security contributions).

Table 4: The fee for the high real estate gains

La taxe sur les plus-values immobilières élevées	
Montant de la PV	Montant de la taxe additionnelle
Source : Direction générale des impôts	
Entre 50 001 € et 60 000 €	2 % x PV - (60 000 - PV) x 1/20
Entre 60 001 € et 100 000 €	2% x PV
Entre 100 001 € et 110 000 €	3% x PV - (110 000 - PV) x 1/10
Entre 110 001 € et 150 000 €	3% x PV
Entre 150 001 € et 160 000 €	4% x PV - (160 000 - PV) x 15/100
Entre 160 001 € et 200 000 €	4% x PV
Entre 200 001 € et 210 000 €	5% x PV - (210 000 - PV) x 20/100
Entre 210 001 € et 250 000 €	5% x PV
Entre 250 001 € et 260 000 €	6% x PV - (260 000 - PV) x 25/100
Supérieur à 260 000 €	6% x PV

The fee for the high real estate gains correspond to the following graphs:



Fee Function	Fee rate fucntion

Figure 23: The fee for the high real estate gains

To take account of fee for the high real estate gains, denote the capital gains by

$$CG_t = P_t - (1 + f_a + f_{cw})P_0 = P_0\left[\mathrm{e}^{\left[(\mu - 0.5\sigma^2)t + \sigma W_t\right]} - (1 + f_a + f_{cw}) \right]$$

We have to compute expectations of the following kind: for constant parameters a_i, b_i and c_i,

$$E\left[\left(a_i CG_t - b_i \right) II_{c_i \le CG_t \le c_{i+1}} \right]$$

Lemma 2. Using the Black and Scholes framework we get an explicit formula for the expectation of the fee for the high real estate gains, which looks like a sum of European options with payoffs defined as follows:

$$e^{-\mu t} E\left[CG_t - b_i II_{c_i \le CG_t \le c_{i+1}} \right] =$$

$$a_i P_0 \left[N(d_{1i}) - N(d_{1(i+1)}) \right] - \left[a_i P_0 \left(1 + f_a + f_{cw} \right) + b_i \right] e^{-\mu t} \left[N(d_{2i}) - N(d_{2(i+1)}) \right]$$

$$\begin{cases} d_{1i} = \dfrac{-Log\left(c_i / P_0 + \left(1 + f_a + f_{cw} \right) \right) + (\mu + 0.5\sigma^2)t}{\sigma\sqrt{t}} \\ \qquad\qquad d_{2i} = d_{1i} - \sigma\sqrt{t} \end{cases}$$